Introduction to Analysis – MA 511 – Fall 2018 – R. Pollack
HW #4

Questions from Rudin:
Chapter 2 – 1,4,9,10,11

1) To see that the empty set is contained in an arbitrary set A, we need to check that every element of the empty set is in A. Done! The empty set has no elements.

4) The set of irrational numbers $\mathbb{I}$ is uncountable. Indeed, if $\mathbb{I}$ were countable, then $\mathbb{I} \cup \mathbb{Q}$ would be countable as the union of countable sets is countable (Theorem 2.12). But $\mathbb{I} \cup \mathbb{Q} = \mathbb{R}$ and $\mathbb{R}$ is uncountable.

9a) Let $x \in E^\circ$. We need to find an open ball centered at x which is entirely contained in E°. By definition of E°, there exists some $r > 0$ such that $N_r(x) \subseteq E$. We claim that $N_r(x) \subseteq E^\circ$. To see this, let $y \in N_r(x)$ and set $h = d(x, y)$. We will check that $N_{r-h}(y) \subseteq N_r(x)$. Indeed, if $z \in N_s(y)$, then

$$d(x, z) \leq d(x, y) + d(y, z) < h + r - h = r,$$

and thus $z \in N_r(x)$. Therefore, $N_{r-h}(y) \subseteq N_r(x)$ and since $N_r(x) \subseteq E$, we deduce that y is an interior point of E, that is $y \in E^\circ$. Since y was arbitrary, we get that $N_r(x) \subseteq E^\circ$ and that E° is open.

9b) This is essentially the definition. We have that E is open if and only if every point of E is an interior point which is true if and only if $E = E^\circ$.

9c) Let $g \in G$. Since G is open, there exists $r > 0$ such that $N_r(g) \subseteq G \subseteq E$. Thus, g is an interior point of E and $G \subseteq E^\circ$.

9d) Since $E^\circ \subseteq E$, we have $(E^\circ)^c \supseteq E^c$. Then by Theorem 2.27c, $(E^\circ)^c \supseteq \overline{E^c}$ as $(E^\circ)^c$ is closed (being the complement of an open set).

To check the reverse inclusion, take $x \in (E^\circ)^c$. So $x \notin E^\circ$. Thus, for every r, we have $N_r(x)$ is not contained in E. That is, there is some $z \in N_r(x)$ such that z is not in E. Thus $z \in E^c$. So we have shown that every neighborhood of $x \in (E^\circ)^c$ contains a point in E^c. This means $(E^\circ)^c \subseteq \overline{E^c}$.

9e) No. Take $E = \mathbb{Q}$. Then $\overline{E}$ equals $\mathbb{R}$. But then E° is empty while $\left(\overline{E}\right)^\circ = \mathbb{R}$.

9f) No. Take $E = \mathbb{Q}$. Then E° is empty and has empty closure while the closure of $\mathbb{Q}$ is $\mathbb{R}$.

10) Property 1 is clear. Symmetry is also clear. For the triangle inequality, we need to check

$$d(x, z) \leq d(x, y) + d(y, z).$$

If $x = z$, this is clear as $d(x, z) = 0$. If $x \neq z$, then either $y \neq x$ or $y \neq z$. In this case, the left hand side is 1 while the right hand side is at least 1.

11) d_1 is not a metric as it fails the triangle inequality. Indeed, $d_1(0, 1/2) = 1/4$ and $d_1(1/2, 1) = 1/4$, but $d_1(0, 1) = 1 \geq 1/4 + 1/4$.

d_2 is a metric. The first property and symmetry are clear. For the triangle inequality, we argue as follows. For any $a, b \in \mathbb{R}$, we have

$$(\sqrt{a} + \sqrt{b})^2 = a + b + 2\sqrt{ab} \geq a + b$$

and thus

$$\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}.$$

Now, by the usual triangle inequality, we have

$$|x - z| \leq |x - y| + |y - z|$$

and thus

$$\sqrt{|x-z|} \leq \sqrt{|x-y|+|y-z|} \leq \sqrt{|x-y|} + \sqrt{|y-z|}$$

where the last inequality is using $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$.

d_3 is not a metric. Indeed $d_3(1,-1) = 0$ but $1 \neq -1$.

d_4 is not a metric. Indeed $d_4(2,1) = 0$ but $2 \neq 1$.

Additional questions:

1. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$, define

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2|.$$

Prove this function (the "taxi-cab" metric) is indeed a metric.

If $d(x,y) = 0$ then both $|x_1 - y_1| = 0$ and $|x_2 - y_2| = 0$. Thus, $x_1 = y_1$, $x_2 = y_2$ and hence $x = y$. Symmetry is clear. Also, for $z = (z_1, z_2)$,

$$d(x,z) = |x_1 - z_1| + |x_2 - z_2| \leq |x_1 - y_1| + |y_1 - z_1| + |x_2 - y_2| + |y_2 - z_2| = d(x,y) + d(y,z)$$

which gives the triangle inequality.

2. Recall that if E is a subset of a metric space, then $\overline{E}$ denotes the closure of E and E° denotes the interior of E. If $E = \mathbb{Q}$ thought of as a subset of $\mathbb{R}$ with the standard metric, determine $\overline{E}$ and E°. Justify your answers.

Solution: We have $\overline{E} = \mathbb{R}$. Indeed, for any $x \in \mathbb{R}$, consider $N_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$. This interval contains infinitely many rationals as proven in class. Thus $x \in \overline{E}$.

We have E° is empty. Indeed, if $x \in E^\circ$, then there is some $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq \mathbb{Q}$. But any such interval is uncountable while $\mathbb{Q}$ is countable.

3. Let A and B be subsets of a metric space. Prove that

$$\overline{A \cup B} = \overline{A} \cup \overline{B},$$

that is, the closure of $A \cup B$ equals the closure of A union the closure of B.

Solution: We repeatedly use the fact that if $X \subseteq F$ with F closed, then $\overline{X} \subseteq F$ (Theorem 2.27c).

Clearly, $A \subseteq A \cup B \subseteq \overline{A \cup B}$. Then since $\overline{A \cup B}$ is closed, by the above fact, $\overline{A} \subseteq \overline{A \cup B}$. Likewise, $\overline{B} \subseteq \overline{A \cup B}$, and thus $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

Converse, since both $\overline{A}$ and $\overline{B}$ are closed, $\overline{A} \cup \overline{B}$ is closed. Then since $A \cup B \subseteq \overline{A} \cup \overline{B}$, we have $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

Since we have proven both containments, equality follows.

4. Let A_1, A_2, ... be subsets of a metric space. Is it true that

$$\overline{\bigcup_{i=1}^{\infty} A_i} = \bigcup_{i=1}^{\infty} \overline{A_i}?$$

If so, prove it. If not, give a counter-example, and point out what part of your proof of #2 breaks down.

Solution: This is false. Consider $A_i = (-1 + 1/i, 1 - 1/i)$. Then

$$\overline{\bigcup_{i=1}^{\infty} A_i} = \overline{\bigcup_{i=1}^{\infty} (-1 + 1/i, 1 - 1/i)} = \overline{(-1,1)} = [-1,1],$$

while

$$\bigcup_{i=1}^{\infty} \overline{A_i} = \bigcup_{i=1}^{\infty} \overline{(-1+1/i, 1-1/i)} = \bigcup_{i=1}^{\infty} [-1+1/i, 1-1/i] = (-1,1).$$

The above proof uses the fact that the union of two closed sets is closed. However, the union of infinitely many closed sets need not be closed.