Introduction to Analysis – MA 511 – Fall 2011 – R. Pollack
HW #5

Questions from Rudin:
Chapter 2 – 12
Chapter 3 – 1

Additional questions:

1. (a) Let $F_1, F_2, \ldots, F_n$ be closed subsets of a metric space. Prove that $\cap_{i=1}^n F_i$ and $\cup_{i=1}^n F_i$ are closed sets directly from the definition of closed (i.e. do not use the criteria F is closed iff F^c is open).
   (b) Does your proof for the previous part extend to infinite unions? to infinite intersections? If not, give a counter-example.

2. Let $X_1 = \mathbb{R}^2$ denote the metric space where $\mathbb{R}^2$ is endowed with the metric $d(x,y) = ||x - y||$ (the standard metric). Let $X_2 = \mathbb{R}^2$ denote the metric space where $\mathbb{R}^2$ is endowed with the metric $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$ (the taxi-cab metric).

   Let $x \in \mathbb{R}^2$. Let $N_r(x)$ denote the ball of radius of r around x in X_1 (i.e. with respect to the standard metric). Let $\hat{N}_r(x)$ denote the ball of radius of r around x in X_2 (i.e. with respect to the taxi-cab metrix).

   (a) For each $r \in \mathbb{R}^{>0}$, show that there exists an $s \in \mathbb{R}^{>0}$ such that $\hat{N}_s(x) \subseteq N_r(x)$.
   (b) For each $r \in \mathbb{R}^{>0}$, show that there exists an $s \in \mathbb{R}^{>0}$ such that $N_s(x) \subseteq \hat{N}_r(x)$.
   (c) Let $U \subseteq \mathbb{R}^2$. Prove that U is open in the standard metric if and only if U is open in the taxi-cab metric.
   (d) Prove $p_n \to p$ in the standard metric if and only if $p_n \to p$ in the taxi-cab metric.

3. Prove each of the following statements directly from the definition of a convergent sequence.
   (a) If $p_n = \frac{1}{n^3}$, then $p_n \to 0$ in $\mathbb{R}$ under the standard metric.
   (b) If $p_n = \frac{2n-1}{3n+2}$, then $p_n \to \frac{2}{3}$ in $\mathbb{R}$ under the standard metric.
   (c) If $p_n = (\frac{1}{n^3}, \frac{2n-1}{3n+2})$, then $p_n \to (0, \frac{2}{3})$ in $\mathbb{R}^2$ under the standard metric.

4. Let X be a metric space with the discrete metric. If $p_n \to p$, what can you say about the sequence $\{p_n\}$?

5. Let X be a metric space and let $x \in X$. Define the closed ball of radius r around x as
$$\overline{N}_r(x) = \{z \in X \mid d(z,x) \leq r\}.$$
   Prove that $\overline{N}_r(x)$ is a closed set.

6. Let K_1 and K_2 be compact subsets of a metric space X. Prove that $K_1 \cup K_2$ is compact and that $K_1 \cap K_2$ is compact. Are these statements still true is we instead consider finite unions or finite intersections of compact sets? What happens if we consider infinite unions or infinite intersections of compact sets? Completely justify your answers with either proofs or counter-examples!

7. Give an example of each of the following or prove that no such example exists.
   (a) A subset E of $\mathbb{R}^2$ such that both E and E^c are neither open nor closed.
   (b) A subset E of $\mathbb{R}^2$ such that both E and E^c are compact.
   (c) A subset E of $\mathbb{R}^2$ such that E is bounded and E^c is closed.
   (d) A convergent sequence $\{s_n\}$ in $\mathbb{R}$ that is unbounded.
   (e) A Cauchy sequence $\{s_n\}$ in $\mathbb{R}$ that is unbounded.