Introduction to Analysis – MA 511 – Fall 2018 – R. Pollack
HW #5 Solutions

Questions from Rudin:
Chapter 2 – 12,15

Solution: (#12) Let $\{U_\alpha\}$ be an open cover of K. Thus there is some U_β containing 0. But since $\{\frac{1}{n}\}$ converges to 0 and U_β is open, all but finitely many fractions $\frac{1}{n}$ are in U_β. Each of these finitely many exceptions are in some open set in our open cover, say $U_{\alpha_1}, \dots, U_{\alpha_n}$. Then $U_{\alpha_1}, \dots, U_{\alpha_n}$ together with U_β gives a finite subcover of K.

Chapter 3 – 1

Solution: (#1) We begin with a lemma.

Lemma 0.1. *For $a, b \in \mathbb{R}^k$, we have*

$$\Big| ||b|| - ||a|| \Big| \leq ||b - a||.$$

Proof. By the triangle inequality,

$$||b|| = ||b - a + a|| \leq ||b - a|| + ||a||$$

and thus

$$||b|| - ||a|| \leq ||b - a||.$$

Arguing in the same way with a and b reversed, we get

$$||a|| - ||b|| \leq ||b - a||,$$

and thus

$$\Big| ||b|| - ||a|| \Big| \leq ||b - a||.$$

□

Now returning to question (#1), we are given that $\{s_n\}$ converges to some element of $\mathbb{R}^k$, say s. We claim that $\{||s_n||\}$ converges to $||s||$. To see this, fix $\varepsilon > 0$. Then, by the convergence of $\{s_n\}$, there exists N such that $n > N$ implies $||s_n - s|| < \varepsilon$. Now, by the lemma, we have

$$\Big| ||s_n|| - ||s|| \Big| \leq ||s_n - s||$$

and thus for $n > N$, we have $\Big| ||s_n|| - ||s|| \Big| \leq ||s_n - s|| < \varepsilon$ which proves that $||s_n|| \to ||s||$.

Additional questions:

1. Let F_1, F_2, $\ldots$, F_n be closed subsets of a metric space. Prove that $\cap_{i=1}^n F_i$ and $\cup_{i=1}^n F_i$ are closed sets directly from the definition of closed (i.e. do not use the criteria F is closed iff F^c is open).

   Solution: Let x be a limit point of $\cap_{i=1}^n F_i$. Thus, every neighborhood of x contains a point z in $\cap_{i=1}^n F_i$ different from x. Thus, z is in each F_i which means that x is a limit point of F_i. Since F_i is closed, $x \in F_i$ for each i. Hence, $x \in \cap_{i=1}^n F_i$ which implies that $\cap_{i=1}^n F_i$ is closed.

   Let x be a limit point of $\cup_{i=1}^n F_i$. Assume x is not a limit point of any of the F_i. Then for each i, there exists a radius $r_i \in \mathbb{R}^{>0}$ such that $N_{r_i}(x)$ contains no point in F_i other than perhaps x. But then if $r = \min\{r_1, \ldots, r_n\}$, we would have that $N_r(x)$ contains no point of $\cup_{i=1}^n F_i$ other than perhaps x. This contradicts that x is a limit point of $\cup_{i=1}^n F_i$. Hence, x is a limit point of some F_i. Since F_i is closed, then $x \in F_i$ and hence $x \in \cup_{i=1}^n F_i$. Therefore, $\cup_{i=1}^n F_i$ is closed.

2. Let $X_1 = \mathbb{R}^2$ denote the metric space where $\mathbb{R}^2$ is endowed with the metric $d(x, y) = ||x - y||$ (the standard metric). Let $X_2 = \mathbb{R}^2$ denote the metric space where $\mathbb{R}^2$ is endowed with the metric $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$ (the taxi-cab metric).

   Let $x \in \mathbb{R}^2$. Let $N_r(x)$ denote the ball of radius of r around x in X_1 (i.e. with respect to the standard metric). Let $\hat{N}_r(x)$ denote the ball of radius of r around x in X_2 (i.e. with respect to the taxi-cab metrix).

   (a) For each $r \in \mathbb{R}^{>0}$, show that there exists an $s \in \mathbb{R}^{>0}$ such that $\hat{N}_s(x) \subseteq N_r(x)$.

   Solution: It's easy to check that $\hat{N}_r(x) \subseteq N_r(x)$.

   (b) For each $r \in \mathbb{R}^{>0}$, show that there exists an $s \in \mathbb{R}^{>0}$ such that $N_s(x) \subseteq \hat{N}_r(x)$.

   Solution: One checks that $N_{\frac{\sqrt{2}r}{2}}(x) \subseteq \hat{N}_r(x)$.

   (c) Let $U \subseteq \mathbb{R}^2$. Prove that U is open in the standard metric if and only if U is open in the taxi-cab metric.

   Solution: Assume U is open in the standard metric. Then for $x \in U$, there exists r such that $N_r(x) \subseteq U$. But then by the above exercise, there exists s such that $\hat{N}_s(x) \subseteq N_r(x)$. Hence, $\hat{N}_s(x) \subseteq U$ and U is open in the taxicab metric. The converse is proven in exactly the same way.

   (d) Prove $p_n \to p$ in the standard metric if and only if $p_n \to p$ in the taxi-cab metric.

   Solution: We know that in any metric space $p_n \to p$ if and only if for every open U all but finitely many p_n are in U. Thus convergence in a metric space is completely determined by the open sets. Since the open sets in the standard metric are the same as open sets in the taxi-cab metric, we deduce that $p_n \to p$ in the standard metric if and only if $p_n \to p$ in the taxi-cab metric.

3. Prove each of the following statements directly from the definition of a convergent sequence.

   (a) If $p_n = \frac{1}{n^3}$, then $p_n \to 0$ in $\mathbb{R}$ under the standard metric.

   Solution: Fix $\varepsilon > 0$ and let $N = \left(\frac{1}{\varepsilon}\right)^{1/3}$. Then

   $$n > N \implies n > \left(\frac{1}{\varepsilon}\right)^{1/3} \implies n^3 > \frac{1}{\varepsilon} \implies \frac{1}{n^3} < \varepsilon.$$

   Thus,

   $$\left|\frac{1}{n^3} - 0\right| = \left|\frac{1}{n^3}\right| = \frac{1}{n^3} < \varepsilon,$$

   and we deduce $\frac{1}{n^3} \to 0$.

(b) If $p_n = \frac{2n-1}{3n+2}$, then $p_n \to \frac{2}{3}$ in $\mathbb{R}$ under the standard metric.

Solution: Fix $\varepsilon > 0$ and let $N = \frac{7}{9\varepsilon} - \frac{2}{3}$. Then

$$n > N \implies n > \frac{7}{9\varepsilon} - \frac{2}{3} \implies 3n > \frac{7}{3\varepsilon} - 2 \implies 3n+2 > \frac{7}{3\varepsilon} \implies \frac{3(3n+2)}{7} > \frac{1}{\varepsilon} \implies \frac{7}{3(3n+2)} < \varepsilon.$$

Thus,

$$\left| \frac{2n-1}{3n+2} - \frac{2}{3} \right| = \left| \frac{6n-3-6n-4}{3(3n+2)} \right| = \left| \frac{-7}{3(3n+2)} \right| = \frac{7}{3(3n+2)} < \varepsilon,$$

and hence, $\frac{2n-1}{3n+2} \to \frac{2}{3}$.

(c) If $p_n = (\frac{1}{n^3}, \frac{2n-1}{3n+2})$, then $p_n \to (0, \frac{2}{3})$ in $\mathbb{R}^2$ under the standard metric.

Solution: Fix $\varepsilon > 0$. Since we know by (a) that $\frac{1}{n^3} \to 0$, we can find N_1 such that $n > N_1$ implies

$$\left| \frac{1}{n^3} - 0 \right| < \frac{\varepsilon}{\sqrt{2}}.$$

Similarly, by (b), we can find N_2 such that for $n > N_2$, we have

$$\left| \frac{2n-1}{3n+2} - \frac{2}{3} \right| < \frac{\varepsilon}{\sqrt{2}}.$$

Let $N_1 = \left(\frac{1}{\varepsilon/\sqrt{2}} \right)^{1/3}$ and $N_2 = \frac{7}{9(\varepsilon/\sqrt{2})} - \frac{2}{3}$. By the same computations, as in parts (a) and (b), we get that for $n > N_1$, we have

$$\left| \frac{1}{n^3} - 0 \right| < \frac{\varepsilon}{\sqrt{2}},$$

and for $n > N_2$, we have

$$\left| \frac{2n-1}{3n+2} - \frac{2}{3} \right| < \frac{\varepsilon}{\sqrt{2}}.$$

Set $N = \max\{N_1, N_2\}$. Then for $n > N$ we have

$$\left\| \left(\frac{1}{n^3}, \frac{2n-1}{3n+2} \right) - \left(0, \frac{2}{3} \right) \right\| = \sqrt{\left(\frac{1}{n^3} - 0 \right)^2 + \left(\frac{2n-1}{3n+2} - \frac{2}{3} \right)^2} < \sqrt{\left(\frac{\varepsilon}{\sqrt{2}} \right)^2 + \left(\frac{\varepsilon}{\sqrt{2}} \right)^2} = \varepsilon.$$

Thus, $(\frac{1}{n^3}, \frac{2n-1}{3n+2}) \to (0, \frac{2}{3})$.

4. Let X be a metric space with the discrete metric. If $p_n \to p$, what can you say about the sequence $\{p_n\}$?

Solution: Since $p_n \to p$, for $\varepsilon = \frac{1}{2}$, there is some N such that $n > N$ implies $d(p_n, p) < \frac{1}{2}$. But in the discrete metric, $d(p_n, p) < \frac{1}{2}$ implies $d(p_n, p) = 0$. Thus for n large enough, $p = p_n$. Thus the sequence $\{p_n\}$ eventually stabilizes to p.

5. Let X be a metric space and let $x \in X$. Define the closed ball of radius r around x as

$$\overline{N}_r(x) = \{z \in X \mid d(z, x) \leq r\}.$$

Prove that $\overline{N}_r(x)$ is a closed set.

Solution: We will prove that the complement of $\overline{N}_r(x)$ is open. To this end, take $w \in \overline{N}_r(x)^c$. Thus, $d(w, x) > r$. Set $s = d(w, x) - r$ which is positive, and we claim that $N_s(w)$ is entirely contained in

$\overline{N}_r(x)^c$. To see this, take $y \in N_s(w)$ and assume $y \in \overline{N}_r(x)$ so that $d(x, y) \le r$. Then, by the triangle inequality, we have

$$d(x, w) \le d(x, y) + d(y, w) < r + s = r + d(w, x) - r = d(w, x).$$

But then $d(x, w) < d(x, w)$. This is a contradiction and thus $y \in \overline{N}_r(x)^c$ and $N_s(w) \subseteq \overline{N}_r(x)^c$. This implies that $\overline{N}_r(x)^c$ is open and $\overline{N}_r(x)$ is closed.

6. Let K_1 and K_2 be compact subsets of a metric space X. Prove that $K_1 \cup K_2$ is compact and that $K_1 \cap K_2$ is compact. Are these statements still true is we instead consider finite unions or finite intersections of compact sets? What happens if we consider infinite unions or infinite intersections of compact sets? Completely justify your answers with either proofs or counter-examples!

   Solution: First, we will check that $K_1 \cup K_2$ is compact. To this end, take an open cover $\{U_\alpha\}$ of $K_1 \cup K_2$. But then $\{U_\alpha\}$ is an open cover of K_1 and of K_2. Since K_1 is compact, there exists some $U_{\alpha_1}, \ldots, U_{\alpha_n}$ which cover K_1. Likewise, there exists some $U_{\beta_1}, \ldots, U_{\beta_m}$ which cover K_2. But then $U_{\alpha_1}, \ldots, U_{\alpha_n}, U_{\beta_1}, \ldots, U_{\beta_m}$ gives a finite subcover of $K_1 \cup K_2$.

   This argument easily generalizes to show that any finite union of compact sets is again compact. It is not true however that arbitrary unions of compact sets are again compact. Indeed, $[n, n+1]$ is compact for every n, but the union of all of these sets is $\mathbb{R}$ which is not compact.

   Now, we will check that if $\{K_\alpha\}$ is some arbitrary collection of compact sets then $\cap_\alpha K_\alpha$ is again compact. To see this, first note that by a proposition proven in class each K_α is closed. Thus, $\cap_\alpha K_\alpha$ is closed. But by another proposition in class, any closed subset of a compact set is again compact.

7. Give an example of each of the following or prove that no such example exists.

   (a) A subset E of $\mathbb{R}^2$ such that both E and E^c are neither open nor closed.

   Solution: Let $x = (0, 0)$ and take $E = N_1(x) \cup (1, 1)$ – that is, the open unit disc together with the single point $(1, 1)$. Then E is not open because (1,1) is not an interior point. Also, E is not closed because (0,1) is a limit point not in E. Further E^c is not open since $(0, 1)$ is not an interior point, and E^c is not closed since (1,1) is a limit point not in the set.

   (b) A subset E of $\mathbb{R}^2$ such that both E and E^c are compact.

   Solution: This is impossible. If E is compact, then E is bounded, and if E^c is compact then E^c is bounded. But the union of two bounded sets is again bounded and the union of E and E^c is $\mathbb{R}^2$ which is not bounded.

   (c) A subset E of $\mathbb{R}^2$ such that E is bounded and E^c is closed.

   Solution: Take E equal to the empty set. Then E is vacuously bounded and $E^c = \mathbb{R}^2$ which is closed.

   (d) A convergent sequence $\{s_n\}$ in $\mathbb{R}$ that is unbounded.

   Solution: Impossible as every convergent sequence is Cauchy and Cauchy sequences are bounded – see below.