Introduction to Analysis – MA 511 – Fall 2018 – R. Pollack
HW #6

Questions from Rudin: Chapter 2: 8

Additional questions:

1. If X is a metric space, and $A \subseteq X$ is bounded, show that $\overline{A}$ is bounded.
2. Let $X = \mathbb{R}^2$ under the standard metric. Let $U = N_1((0,0))$ which is the open unit ball of radius 1 and let $F = \overline{N}_1((0,0))$ which is the closed unit ball of radius 1.
   (a) Give an example of an open cover of U which has no finite subcover.
   (b) Give an example of an open cover of U which has a finite subcover.
   (c) By adding more open sets, extend your answer to part (a) to an open cover of F. Exhibit a finite subcover of this open cover (which must exist since F is compact).
3. We make the following important definition: a sequence $\{s_n\}$ is called a *Cauchy sequence* if for every $\varepsilon > 0$ there is an integer N such that whenever $n, m > N$ we have $d(s_n, s_m) < \varepsilon$.
   (a) Prove $\{1/n\}$ is a Cauchy sequence.
   (b) If $\{s_n\}$ and $\{t_n\}$ are Cauchy sequences in $\mathbb{R}$, prove that $\{s_n + t_n\}$ is also a Cauchy sequence.
   (c) We will see in class that every Cauchy sequence is automatically a convergent sequence. Can you find an example of a metric space and a Cauchy sequence in that metric space that is not convergent?
4. (a) Let $\{s_n\}$ and $\{t_n\}$ be two Cauchy sequences in $\mathbb{Q}$. We define a relation on such Cauchy sequences as follows
   $$\{s_n\} \sim \{t_n\} \iff \{s_n - t_n\} \text{ converges to } 0.$$
   Prove that $\sim$ is an equivalence relation.
   (b) Give an example of two Cauchy sequences $\{s_n\}$ and $\{t_n\}$ in $\mathbb{Q}$ such that $\{s_n\} \neq \{t_n\}$ but $\{s_n\} \sim \{t_n\}$.
5. **(Optional challenge problems)** Cauchy sequences are extremely convenient because their definition does not refer to their limit. In fact, a standard use of Cauchy sequences is to construct $\mathbb{R}$ from $\mathbb{Q}$. We outline this construction in this exercise.

   To this end, define the set of real numbers $\mathbb{R}$ to be the set of equivalence classes of Cauchy sequences in $\mathbb{Q}$. That is, if $\{s_n\}$ is a Cauchy sequence in $\mathbb{Q}$, we let $[s_n]$ denote its equivalence class under $\sim$, i.e.
   $$[s_n] = \Big\{ \{t_n\} \text{ such that } \{t_n\} \text{ is a Cauchy sequence in } \mathbb{Q} \text{ and } \{s_n\} \sim \{t_n\} \Big\}.$$
   Then $\mathbb{R}$ is defined to be the collection of all $[s_n]$.

   We now need to define $+$, $\cdot$, $<$, on $\mathbb{R}$ and prove that $\mathbb{R}$ is a complete ordered field satisfying the least upper bound axiom.
   (a) For $\{s_n\}$ and $\{t_n\}$ Cauchy sequences in $\mathbb{Q}$, define $[s_n] + [t_n] = [s_n + t_n]$. (Note that this definition implicitly makes use of exercise 3b.) Prove that this definition is well-defined. That is, prove that if $\{s_n\} \sim \{s_n'\}$ and $\{t_n\} \sim \{t_n'\}$, then $\{s_n + t_n\} \sim \{s_n' + t_n'\}$.
   (b) Define $[s_n] \cdot [t_n] = [s_n \cdot t_n]$ and again prove that this is well-defined.
   (c) Prove that $\mathbb{R}$ is a field under $+$ and $\cdot$.
   (d) Define a relation $\leq$ on $\mathbb{R}$ by $[s_n] \leq [t_n]$ if there exists N such that $n > N$ implies $s_n \leq t_n$. Prove this relation is well-defined.
   (e) Prove that $\mathbb{R}$ is an ordered field under $<$. (Here $[\{s_n\}] < [\{t_n\}]$ if $[\{s_n\}] \leq [\{t_n\}]$ and $[\{s_n\}] \neq [\{t_n\}]$.)
   (f) Prove that $\mathbb{R}$ satisfies the least upper bound axiom.