Introduction to Analysis – MA 511 – Fall 2018 – R. Pollack
HW #6

Questions from Rudin: Chapter 2: 8

Solution: #8: Yes, every point of an open set U of $\mathbb{R}^2$ is a limit point of U. To see this, take $x \in U$. Then since U is open, there exists $r > 0$ such that $N_r(x) \subseteq U$. Now to see that x is a limit point, we need to take an arbitrary $s > 0$ and check that $N_s(x) \cap U$ contains some point other than x. But, if $t = \min\{r, s\}$, then $N_s(x) \cap U = N_t(x)$ which clearly contains infinitely many points of U other than x.

The corresponding statement for closed sets is false. Indeed, a set made up of a single point is closed but has no limit points.

Additional questions:

1. If X is a metric space, and $A \subseteq X$ is bounded, show that $\overline{A}$ is bounded.

   Solution: Since A is bounded we know that there is some $r > 0$ and $x \in X$ such that $A \subseteq N_r(x)$. Thus, $A \subseteq \overline{N}_r(x)$. Since $\overline{N}_r(x)$ is a closed set, we have $\overline{A} \subseteq \overline{N}_r(x)$ (Rudin Thm 2.27c). Since $\overline{N}_r(x)$ is clearly contained in $N_{r+1}(x)$, we have $\overline{A} \subseteq N_{r+1}(x)$, and thus $\overline{A}$ is bounded.

2. Let $X = \mathbb{R}^2$ under the standard metric. Let $U = N_1((0, 0))$ which is the open unit ball of radius 1 and let $F = \overline{N}_1((0, 0))$ which is the closed unit ball of radius 1.

   (a) Give an example of an open cover of U which has no finite subcover.
   Solution: Let $U_n = N_{1-1/n}((0, 0))$. This is an open cover since any $x \in U$ has distance to the origin strictly less than 1 (and thus is in U_n for large enough n). However, any finite union of the U_n equals U_N for some N which is clearly not equal to all U.

   (b) Give an example of an open cover of U which has a finite subcover.
   Solution: Simply take an open cover with one set: $U_1 = U$. This cover of course has a finite subcover, namely itself.

   (c) By adding more open sets, extend your answer to part (a) to an open cover of F. Exhibit a finite subcover of this open cover (which must exist since F is compact).
   Solution: Define an annulus

   $$U_0 = \{x \in \mathbb{R}^2 \ : \ 1/2 < ||x|| < 3/2\}$$

   which is clearly an open set. Then the cover $\{U_n\}$ from (a) together with U_0 is an open cover of F. A finite subcover of this open cover is U_3 and U_0 since U_3 includes all points of length less than $2/3$ while U_0 contains vectors of length greater than $1/2$.

3. We make the following important definition: a sequence $\{s_n\}$ is called a *Cauchy sequence* if for every $\varepsilon > 0$ there is an integer N such that whenever $n, m > N$ we have $d(s_n, s_m) < \varepsilon$.

   (a) Prove $\{1/n\}$ is a Cauchy sequence.
   Solution: Fix $\varepsilon > 0$ and let $N = 1/\varepsilon$. Then take arbitrary $m, n > N$ and without loss of generality assume that $n > m$. (This means that it is fair to assume $n > m$ because the case $n \leq m$ is exactly the same.) Then

   $$\left|\frac{1}{m} - \frac{1}{n}\right| < \left|\frac{1}{m}\right| < \left|\frac{1}{N}\right| < \varepsilon$$

   as desired.

   (b) If $\{s_n\}$ and $\{t_n\}$ are Cauchy sequences in $\mathbb{R}$, prove that $\{s_n + t_n\}$ is also a Cauchy sequence.
   Solution: Fix $\varepsilon > 0$. Then there exists N_1 such that $m, n > N_1$ implies $d(s_n, s_m) < \varepsilon/2$. Likewise, there exists N_2 such that $m, n > N_1$ implies $d(t_n, t_m) < \varepsilon/2$. Set $N = \max\{N_1, N_2\}$. Then for $m, n > N$, we have

   $$||(s_n + t_n) - (s_m + t_m)|| = ||(s_n - s_m) + (t_n - t_m)|| \leq ||s_n - s_m|| + ||t_n - t_m|| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

   Thus, $\{s_n + t_n\}$ is a Cauchy sequence.

(c) We will see in class that every Cauchy sequence is automatically a convergent sequence. Can you find an example of a metric space and a Cauchy sequence in that metric space that is not convergent?

Solution: There are many possible solutions. Here is one: take $X = \mathbb{R} - \{0\}$ with the standard metric. Then $\{1/n\}$ is a Cauchy sequence but is not convergent as its limit is 0 which is not in the space.

4. (a) Let $\{s_n\}$ and $\{t_n\}$ be two Cauchy sequences in $\mathbb{Q}$. We define a relation on such Cauchy sequences as follows

$$\{s_n\} \sim \{t_n\} \iff \{s_n - t_n\} \text{ converges to } 0.$$

Prove that $\sim$ is an equivalence relation.

Solution:

Reflexive: Since $\{s_n - s_n\} = \{0\}$ is just the constant sequence 0 which clearly converges to 0, we have $\{s_n\} \sim \{s_n\}$.

Symmetric: if $\{s_n\} \sim \{t_n\}$, then $\{s_n - t_n\}$ converges to 0. By a limit law, $\{(-1)(s_n - t_n)\}$ converges to 0. Hence, $\{t_n - s_n\}$ converges to 0 and $\{t_n\} \sim \{s_n\}$.

Transitive: if $\{s_n\} \sim \{t_n\}$ and $\{t_n\} \sim \{u_n\}$, then $\{s_n - t_n\}$ and $\{t_n - u_n\}$ both converge to 0. Hence (by a limit law), the sum $\{s_n - t_n + t_n - u_n\}$ converges to 0. But this means that $\{s_n - u_n\}$ converges to 0 or $\{s_n\} \sim \{u_n\}$.

(b) Give an example of two Cauchy sequences $\{s_n\}$ and $\{t_n\}$ in $\mathbb{Q}$ such that $\{s_n\} \neq \{t_n\}$ but $\{s_n\} \sim \{t_n\}$.

Solution: Take $s_n = \frac{1}{n}$ and $t_n = \frac{1}{n^2}$. Then $\{s_n\} \sim \{t_n\}$ since $\lim \frac{1}{n} - \frac{1}{n^2} = 0$, but clearly these sequences are distinct.