Introduction to Analysis – MA 511 – Fall 2011 – R. Pollack
HW #7 Solutions

**Questions from Rudin**: Chapter 3: 6(a,b), 9, 10, 23, 24(a,b)

*Solution*: #6(a) – We have

$$s_n = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \sqrt{i+1} - \sqrt{i} = \sqrt{n+1} - \sqrt{1}.$$

Since $s_n$ is unbounded, this sequence must be divergent, and thus $\sum a_n$ is divergent.

*Solution*: #6(b) – We have

$$a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n} = \frac{\sqrt{n+1}-\sqrt{n}}{n} \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{n+1-n}{n(\sqrt{n+1}+\sqrt{n})} = \frac{1}{n(\sqrt{n+1}+\sqrt{n})}.$$

Thus

$$a_n < \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}.$$

Since $\sum_n \frac{1}{n^{3/2}}$ converges, we have that $\sum a_n$ converges by the comparison test.

*Solution*: #10 – Assume that $\limsup |a_n|^{\frac{1}{n}} < 1$. Then by Rudin 3.17(b) (lemma proven in class), for all but finitely many $n$, we have that $|a_n|^{\frac{1}{n}} < 1$. Thus $|a_n| < 1$ for all but finitely many $n$. Since $a_n \in \mathbb{Z}$, this mean that $a_n = 0$ for all but finitely many $n$. But in the question it is assumed that $a_n$ is non-zero for infinitely many $n$. Thus $\limsup |a_n|^{\frac{1}{n}} \geq 1$ and hence the radius of convergence $R = (\limsup |a_n|^{\frac{1}{n}})^{-1}$ is less than or equal to 1.

*Solution*: For all $n$, we have

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n)$$

and thus

$$d(p_n, q_n) - d(p_m, q_m) \leq d(p_n, p_m) + d(q_m, q_n).$$

Reversing the roles of $m$ and $n$ gives

$$d(p_m, q_m) - d(p_n, q_n) \leq d(p_m, p_n) + d(q_n, q_m)$$

and thus

$$|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_m, q_n).$$

To prove $\{d(p_n, q_n)\}$ is a convergent sequence in $\mathbb{R}$, it suffices to see that $\{d(p_n, q_n)\}$ is a Cauchy sequence (since Cauchy implies convergent in $\mathbb{R}^k$). To this end, fix $\varepsilon > 0$. Since $\{p_n\}$ is Cauchy, there is some $N_1$ such that $m, n > N_1$ implies $d(p_n, p_m) < \varepsilon/2$. Similarly, since $\{q_n\}$ is Cauchy, there is some $N_2$ such that $m, n > N_2$ implies $d(q_n, q_m) < \varepsilon/2$. Set $N = \max\{N_1, N_2\}$. Then

$$|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_m, q_n) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence, $\{d(p_n, q_n)\}$ is a Cauchy sequence and thus a convergent sequence.

*Solution*: #24(a) – Let $\{p_n\}$ be a Cauchy sequence. Since $d(p_n, p_n) = 0$ for all $n$, we have that $\lim_n d(p_n, p_n) = 0$ and thus the relation is reflexive.

Now let $\{p_n\}$ and $\{q_n\}$ be two Cauchy sequence such that $\lim_n d(p_n, q_n) = 0$. Since $d(p_n, q_n) = d(q_n, p_n)$, we have

$$\lim_n d(q_n, p_n) = \lim_n d(q_n, p_n) = 0$$

and thus the relation is symmetric.

Now let $\{p_n\}$, $\{q_n\}$ and $\{r_n\}$ be three Cauchy sequence such that $\lim_n d(p_n, q_n) = 0$ and $\lim_n d(q_n, r_n) = 0$. By the triangle inequality, we have $0 \le d(p_n, r_n) \le d(p_n, q_n) + d(q_n, r_n)$ and thus

$$0 \le \lim_n d(p_n, r_n) \le \lim_n d(p_n, q_n) + d(q_n, r_n) = \lim_n d(p_n, q_n) + \lim_n d(q_n, r_n) = 0 + 0 = 0.$$

Hence $\lim_n d(p_n, r_n) = 0$, and the relation is transitive.

*Solution*: #24(b) – We must check that if $\{p_n\}$, $\{p'_n\}$, $\{q_n\}$, $\{q'_n\}$ are Cauchy sequences such that

$$\lim_n d(p_n, p'_n) = 0 \text{ and } \lim_n d(q_n, q'_n) = 0$$

then

$$\lim_n d(p_n, q_n) = \lim_n d(p'_n, q'_n)$$

or equivalently that

$$\lim_n d(p_n, q_n) - d(p'_n, q'_n) = 0.$$

We have

$$d(p_n, q_n) \le d(p_n, p'_n) + d(p'_n, q'_n) + d(q'_n, q_n)$$

and thus

$$d(p_n, q_n) - d(p'_n, q'_n) \le d(p_n, p'_n) + d(q'_n, q_n).$$

Also,

$$d(p'_n, q'_n) \le d(p'_n, p_n) + d(p_n, q_n) + d(q_n, q'_n)$$

and thus

$$d(p'_n, q'_n) - d(p_n, q_n) \le d(p'_n, p_n) + d(q_n, q'_n).$$

Hence,

$$|d(p_n, q_n) - d(p'_n, q'_n)| \le d(p_n, p'_n) + d(q_n, q'_n).$$

Now fix $\varepsilon > 0$. Then there is some $N_1$ such $n > N_1$ implies $d(p_n, p'_n) < \varepsilon/2$. Also, there is some $N_2$ such $n > N_2$ implies $d(q_n, q'_n) < \varepsilon/2$. Set $N = \max\{N_1, N_2\}$. Then for $n > N$, we have

$$|d(p_n, q_n) - d(p'_n, q'_n)| \le d(p_n, p'_n) + d(q_n, q'_n) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

and thus $\lim_n d(p_n, q_n) - d(p'_n, q'_n) = 0$ as desired. Thus, $\Delta(P, Q)$ is well-defined.

We must also check that $\Delta$ satisfies the axioms of a metric space.

Clearly $\Delta(P, P) = 0$. Conversely, let $P, Q \in X^*$ such that $\Delta(P, Q) = 0$. Let $\{p_n\}$ be a representative Cauchy sequence from $P$ and $\{q_n\}$ a representative Cauchy sequence from $Q$. Then $\lim_n d(p_n, q_n) = 0$. Thus by definition $\{p_n\}$ is equivalent to $\{q_n\}$, and again by definition this means that $P = Q$ in $X^*$.

Now we check $\Delta(P, Q) = \Delta(Q, P)$. Let $\{p_n\}$ and $\{q_n\}$ be representatives of $P$ and $Q$ respectively. Then

$$\Delta(P, Q) = \lim_n d(p_n, q_n) = \lim_n d(q_n, p_n) = \Delta(Q, P)$$

as desired.

Now we must check the triangle inequality. So let $P, Q, R \in X^*$ with representatives $\{p_n\}, \{q_n\}$, and $\{r_n\}$. Then

$$\Delta(P, R) = \lim_n d(p_n, r_n) \le \lim_n d(p_n, q_n) + d(q_n, r_n) = \lim_n d(p_n, q_n) + \lim_n d(q_n, r_n) = \Delta(P, Q) + \Delta(Q, R)$$

as desired.

**Additional questions:**

1. Let $f : X \to Y$ be a function between two sets. For a subset $A \subseteq X$, we define

$$f(A) = \{b \in B \text{ such that } b = f(a) \text{ for some } a \in A\},$$

and for a subset $B \subseteq Y$, we define

$$f^{-1}(B) = \{a \in A \text{ such that } f(a) \in B\}.$$

In each of the following pairs of sets, the two sets are related by either $=$, $\subset$ or $\supset$. Determine which is the correct relation and prove your answer. If you answer $\subset$ or $\supset$, give an explicit example where the reverse inclusion does not hold.

(a) $f(A_1 \cup A_2) \longleftrightarrow f(A_1) \cup f(A_2)$
*Solution*: We have $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$. Indeed

$$\begin{aligned} x \in f(A_1 \cup A_2) &\iff x = f(a) \text{ for some } a \in A_1 \cup A_2 \\ &\iff x = f(a) \text{ for some } a \in A_1 \text{ or } a \in A_2 \\ &\iff x \in f(A_1) \text{ or } x \in f(A_2) \\ &\iff x \in f(A_1) \cup f(A_2) \end{aligned}$$

(b) $f(A_1 \cap A_2) \longleftrightarrow f(A_1) \cap f(A_2)$
*Solution*: We have $f(A_1 \cap A_2) \subseteq f(A_1) \cup f(A_2)$. Indeed

$$\begin{aligned} x \in f(A_1 \cap A_2) &\implies x = f(a) \text{ for some } a \in A_1 \cap A_2 \\ &\implies x = f(a) \text{ for some } a \in A_1 \text{ and } a \in A_2 \\ &\implies x \in f(A_1) \text{ and } x \in f(A_2) \\ &\implies x \in f(A_1) \cap f(A_2). \end{aligned}$$

The inclusion $\supseteq$ is not true in general. Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. Then take $A_1 = (-1, 0)$, $A_2 = (0, 1)$. Then $A_1 \cap A_2$ is empty and so $f(A_1 \cap A_2)$ is empty. However, $f(A_1) = (0, 1) = f(A_2)$ and thus $f(A_1) \cap f(A_2) = (0, 1)$.

(c) $f^{-1}(B_1 \cup B_2) \longleftrightarrow f^{-1}(B_1) \cup f^{-1}(B_2)$
*Solution*: We have $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$. Indeed

$$\begin{aligned} x \in f^{-1}(B_1 \cup B_2) &\iff f(x) \in B_1 \cup B_2 \\ &\iff f(x) \in B_1 \text{ or } f(x) \in B_2 \\ &\iff x \in f^{-1}(B_1) \text{ or } x \in f^{-1}(B_2) \\ &\iff x \in f^{-1}(B_1) \cup f^{-1}(B_2) \end{aligned}$$

(d) $f^{-1}(B_1 \cap B_2) \longleftrightarrow f^{-1}(B_1) \cap f^{-1}(B_2)$
*Solution*: We have $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$. Indeed

$$\begin{aligned} x \in f^{-1}(B_1 \cap B_2) &\iff f(x) \in B_1 \cap B_2 \\ &\iff f(x) \in B_1 \text{ and } f(x) \in B_2 \\ &\iff x \in f^{-1}(B_1) \text{ and } x \in f^{-1}(B_2) \\ &\iff x \in f^{-1}(B_1) \cap f^{-1}(B_2) \end{aligned}$$

(e) $f(f^{-1}(B)) \longleftrightarrow B$

*Solution*: We have $f(f^{-1}(B)) \subseteq B$. Indeed, $x \in f(f^{-1}(B))$ implies that $x = f(a)$ for some $a \in f^{-1}(B)$. But then by definition $f(a) \in B$ and thus $x \in B$.

The reverse inclusion need not hold. Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ and take $B = \mathbb{R}$. Then $f^{-1}(B) = f^{-1}(\mathbb{R}) = \mathbb{R}$, and thus $f(f^{-1}(B)) = f(\mathbb{R}) = \mathbb{R}^{\geq 0}$ as $x^2$ is always non-negative.

(f) $f^{-1}(f(A)) \longleftrightarrow A$

*Solution*: We have $f^{-1}(f(A)) \supseteq A$. Indeed, for $x \in A$, we have $f(x) \in f(A)$ and thus $x \in f^{-1}(f(A))$.

The reverse inclusion need not hold. Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ and take $A = \mathbb{R}^{\geq 0}$. Then $f(A) = \mathbb{R}^{\geq 0}$, and thus $f^{-1}(f(A)) = f^{-1}(\mathbb{R}^{\geq 0}) = \mathbb{R}$ as $f(x) = x^2$ is always non-negative.

2. Let $\{a_n\}$ be a convergent sequence in $\mathbb{R}$. Prove that

$$\limsup\{a_n\} = \liminf\{a_n\} = \lim\{a_n\}.$$

*Solution*: We start with a lemma.

**Lemma 0.1.** *Let $\{s_n\}$ denote a convergent sequence in $\mathbb{R}$ with $\{s_n\} \to s$. Then any subsequence of $\{s_n\}$ also converges to $s$.*

*Proof.* Let $\{s_{n_j}\}$ denote a subsequence. Fix $\varepsilon > 0$. Then there exists $N$ such that $n > N$ implies $|s_n - s| < \varepsilon$. Then for $j > N$, we have $n_j > N$ as $n_j \geq j$. Thus $|s_{n_j} - s| < \varepsilon$ and we deduce that $\{s_{n_j}\} \to s$. □

Returning to the question at hand. Let $\{a_n\} \to a$. Then if $E$ denotes the subset of $\mathbb{R}$ consisting of limits of subsequences of $\{a_n\}$, by the lemma, $E = \{a\}$. Thus

$$\limsup\{a_n\} = \sup E = a = \lim\{a_n\}$$

and

$$\liminf\{a_n\} = \inf E = a = \lim\{a_n\}.$$