Introduction to Analysis – MA 511 – Fall 2018 – R. Pollack
HW #8

Determine if each of the below statements are true or false. For the true statements give a proof of why the statement is true. For the false statements, give an explicit counter-example.

You will need to know the following facts which were not proven, but only discussed, in class. (Both are standard second semester calculus facts.)

- Fact: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$.
- Fact: If $a_n > 0$ for each n and $a_n > a_{n+1}$, then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges.

In the below questions, $\{a_n\}$ denotes a sequence of real numbers. The notation $\sum a_n$ means $\displaystyle\sum_{n=1}^{\infty} a_n$.

TRUE/FALSE

1. If $\{a_n\}$ is convergent, then $\sum a_n$ is convergent.
2. If $\sum a_n$ is convergent, then $\{a_n\}$ is convergent.
3. If $\sum a_n$ converges, then $\sum a_n^2$ converges,
4. If $\sum a_n$ diverges, then $\sum a_n^2$ diverges,
5. If each $a_n \geq 0$ and $\sum a_n$ converges, then $\sum a_n^2$ converges,
6. If each $a_n \geq 0$ and $\sum a_n$ diverges, then $\sum a_n^2$ diverges,
7. If $\sum a_n$ converges and $\sum b_n$ converges, then $\sum a_n + b_n$ converges.
8. If $\sum a_n$ converges and $\sum b_n$ converges, then $\sum a_n \cdot b_n$ converges.
9. If r is an irrational number (i.e. r is in $\mathbb{R}$ but not $\mathbb{Q}$), then $\displaystyle\sum_{n=0}^{\infty} r^n$ is never in $\mathbb{Q}$.
10. (Challenge — ungraded) If for every convergent series $\sum b_n$ we have $\sum a_n b_n$ converges, then $\sum a_n$ converges.
11. If $\displaystyle\limsup_{n\to\infty} a_n = \alpha$, then for every $\varepsilon > 0$, there exists infinitely many a_n such that $\alpha - \varepsilon < a_n < \alpha$.
12. For two sequences $\{a_n\}$ and $\{b_n\}$, we have $\displaystyle\limsup_{n\to\infty} a_n + b_n = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.
13. Let $\{a_n\}$ be some sequence with $\displaystyle\limsup_{n\to\infty} a_n = 5$. Define a sequence $\{b_n\}$ by $b_n = \begin{cases} 10 & n \text{ is even} \\ a_n & n \text{ is odd} \end{cases}$. Then $\displaystyle\limsup_{n\to\infty} b_n = 10$.
14. Let $\{a_n\}$ be some sequence with $\displaystyle\limsup_{n\to\infty} a_n = 5$. Define a sequence $\{b_n\}$ by $b_n = \begin{cases} 3 & n \text{ is even} \\ a_n & n \text{ is odd} \end{cases}$. Then $\displaystyle\limsup_{n\to\infty} b_n = 5$.