Introduction to Analysis – MA 511 – Fall 2018 – R. Pollack
HW #8

Determine if each of the below statements are true or false. For the true statements give a proof of why the statement is true. For the false statements, give an explicit counter-example.

You will need to know the following facts which were not proven, but only discussed, in class. (Both are standard second semester calculus facts.)

- Fact: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$.
- Fact: If $a_n > 0$ for each n, $a_n > a_{n+1}$, and $\{a_n\} \to 0$, then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges.

In the below questions, $\{a_n\}$ denotes a sequence of real numbers. The notation $\sum a_n$ means $\displaystyle\sum_{n=1}^{\infty} a_n$.

TRUE/FALSE

1. If $\{a_n\}$ is convergent, then $\sum a_n$ is convergent.

   Solution: False! Consider $a_n = 1$ for all n. Then $\{a_n\}$ is clearly convergent, but the partial sum $s_n = \displaystyle\sum_{k=1}^{n} 1 = n$ which does not converge.

2. If $\sum a_n$ is convergent, then $\{a_n\}$ is convergent.

   Solution: True! If $\sum a_n$ converges, then $\{a_n\} \to 0$ as proven in class.

3. If $\sum a_n$ converges, then $\sum a_n^2$ converges,

   Solution: False! Consider $a_n = (-1)^n \frac{1}{n}$. Then $\sum a_n = \sum(-1)^n \frac{1}{n}$ is convergent (alternating series), but $\sum a_n^2 = \sum \frac{1}{n^2}$ which is convergent.

4. If $\sum a_n$ diverges, then $\sum a_n^2$ diverges,

   Solution: False! Consider $a_n = \frac{1}{n}$. Then $\sum a_n = \sum \frac{1}{n}$ diverges, but $\sum a_n = \sum \frac{1}{n^2}$ converges.

5. If each $a_n \geq 0$ and $\sum a_n$ converges, then $\sum a_n^2$ converges,

   Solution: True! Since $\sum a_n$ converges, we have $\{a_n\} \to 0$. In particular, for n large enough, $a_n < 1$. (Formally, there exists an N such that $n \geq N$ implies $a_n < 1$ which comes from the definition of the limit $\{a_n\} \to 0$.). Thus, $a_n^2 < a_n$ for n large enough. (Here we are using that $a_n > 0$.) Thus, by the comparison test, if $\sum a_n$ converges we must have that $\sum a_n^2$ converges.

6. If each $a_n \geq 0$ and $\sum a_n$ diverges, then $\sum a_n^2$ diverges,

   Solution: False! The counter-example in #4 works here as well.

7. If $\sum a_n$ converges and $\sum b_n$ converges, then $\sum a_n + b_n$ converges.

   Solution: True! Let $\{s_n\}$ denote the partial sums associated to the $\{a_n\}$. That is, $s_n = a_1 + \cdots + a_n$. Likewise let $\{t_n\}$ denote the partial sums associated to the $\{b_n\}$. That is, $t_n = b_1 + \cdots + b_n$. Note then that

   $$s_n + t_n = a_1 + \cdots + a_n + b_1 + \cdots + b_n = a_1 + b_1 + \cdots + a_n + b_n$$

and thus $\{s_n + t_n\}$ is the sequence of partial sums attached to $\{a_n + b_n\}$. Thus, to determine is $\sum a_n + b_n$ converges we need to see that $\{s_n + t_n\}$ converges.

Since $\sum a_n$ converges, by definition $\{s_n\}$ converges. Since $\sum b_n$ converges, by definition $\{t_n\}$ converges. Then, as proven in class, $\{s_n + t_n\}$ converges. Thus $\sum a_n + b_n$ converges which completes the proof.

8. If $\sum a_n$ converges and $\sum b_n$ converges, then $\sum a_n \cdot b_n$ converges.

   Solution: False! Take $a_n = b_n = (-1)^n \frac{1}{\sqrt{n}}$. Then $\sum a_n$ and $\sum b_n$ both converge (alternating series), but $\sum a_n b_n = \sum \frac{1}{n}$ which diverges.

9. If r is an irrational number (i.e. r is in $\mathbb{R}$ but not $\mathbb{Q}$), then $\displaystyle\sum_{n=0}^{\infty} r^n$ is never in $\mathbb{Q}$.

   Solution: True! Write $\alpha = \displaystyle\sum_{n=0}^{\infty} r^n$ so that $\alpha = \frac{1}{1-r}$. Thus, $r = 1 - \frac{1}{\alpha}$. In particular, α is rational, then r must be rational as well. This is the contrapositive of the statement and so we are done.

10. (Challenge — ungraded) If for every convergent series $\sum b_n$ we have $\sum a_n b_n$ converges, then $\sum a_n$ converges.

    Solution: This one turned out to be easy. It is false. Consider $a_n = 1$. Let b_n be any convergent sequence. Then clearly $\sum a_n b_n = \sum b_n$ converges. However, $\sum a_n = \sum 1$ diverges.

11. If $\limsup\limits_{n\to\infty} a_n = \alpha$, then for every $\varepsilon > 0$, there exists infinitely many a_n such that $\alpha - \varepsilon \leq a_n \leq \alpha$.

    Solution: False! Take $a_n = 1 + \frac{1}{n}$. Then $\limsup\limits_{n\to\infty} a_n = 1$ but a_n is never less than 1.

12. For two sequences $\{a_n\}$ and $\{b_n\}$, we have $\limsup\limits_{n\to\infty} a_n + b_n = \limsup\limits_{n\to\infty} a_n + \limsup\limits_{n\to\infty} b_n$.

    Solution: False! Take $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$. Then $\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} b_n = 1$. However, $a_n + b_n = (-1)^n + (-1)^{n+1} = 0$ and thus $\limsup_{n\to\infty} a_n + b_n = 0$.

13. Let $\{a_n\}$ be some sequence with $\limsup\limits_{n\to\infty} a_n = 5$. Define a sequence $\{b_n\}$ by $b_n = \begin{cases} 10 & n \text{ is even} \\ a_n & n \text{ is odd} \end{cases}$. Then $\limsup\limits_{n\to\infty} b_n = 10$.

    Solution: True! For $\varepsilon = 1$, there is some N such that if $n > N$ then $|a_n - 5| < 1$. Thus, for $n > N$, we have $|a_n| < 6$. From this it is clear that $\limsup_{n\to\infty} b_n = 10$ as the a_n's do not affect this lim sup as each a_n is no more than 6.

14. Let $\{a_n\}$ be some sequence with $\limsup\limits_{n\to\infty} a_n = 5$. Define a sequence $\{b_n\}$ by $b_n = \begin{cases} 3 & n \text{ is even} \\ a_n & n \text{ is odd} \end{cases}$. Then $\limsup\limits_{n\to\infty} b_n = 5$.

    Solution: False! Take $a_n = \begin{cases} 5 & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases}$. Then $\limsup\limits_{n\to\infty} a_n = 5$ while $\limsup\limits_{n\to\infty} b_n = 3$.