Introduction to Analysis – MA 511 – Fall 2018 – R. Pollack
HW #9

Questions from Rudin:
Chapter 4: 3,4,18 (replace "has a simple discontinuity" with "is not continuous")

Additional questions:

1. Define a function $f : \mathbb{R} \to \mathbb{R}$ as follows:
$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}.$$
Prove that f is not continuous at any point $p \in \mathbb{R}$.

2. Prove that $f : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ given by $f(x) = \sqrt{x}$ is continuous at all $p > 0$.
Hint: $(\sqrt{x} - \sqrt{p})(\sqrt{x} + \sqrt{p}) = x - p$.

3. Let $f : X \to Y$ be a continuous function between two metric spaces. For each of the following statements, either prove the statement or give some explicit counter-example.
   (a) If $F \subseteq X$ is closed, then $f(F)$ is closed.
   (b) If $U \subseteq X$ is open, then $f(U)$ is open.
   (c) If $B \subseteq X$ is bounded, then $f(B)$ is bounded.

4. Let $f : X \to Y$ be a continuous function between two metric spaces. For each of the following statements, either prove the statement or give some explicit counter-example.
   (a) If $F \subseteq Y$ is closed, then $f^{-1}(F)$ is closed.
   (b) If $B \subseteq Y$ is bounded, then $f^{-1}(B)$ is bounded.
   (c) If $K \subseteq Y$ is compact, then $f^{-1}(K)$ is compact.

   (Hint: Show that $(\inf I, \sup I) \subseteq I$ and then take cases on whether or not $\inf I$ and $\sup I$ are actually in I.)

5. Define a function $f : \mathbb{R} \to \mathbb{R}$ by
$$f(x) = \begin{cases} x \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$
Prove that $f(x)$ is continuous at $x = 0$.
(Hint: You may assume that $|\sin(y)| \leq 1$ for all $y \in \mathbb{R}$.)