Introduction to Analysis – MA 511 – Fall 2018 – R. Pollack
HW #9 solutions

Questions from Rudin:

Chapter 4: 3,4,18 (replace "has a simple discontinuity" with "is not continuous")

Solution: #3: We have $Z(f) = f^{-1}(\{0\})$. Since $\{0\}$ is closed we have $f^{-1}(\{0\})$ and thus $Z(f)$ is closed by 4(a) below. (Slick, no?)

One could also argue directly: namely, show that $Z(f)$ is closed by showing that $Z(f)^c$ is open. Note that $Z(f)^c = \{x \in X \ : \ f(x) \neq 0\}$. Take $x \in Z(f)^c$ and we must find a ball around x completely contained in $Z(f)^c$. That is, we must show that if a function does not vanish at a point, then it does not vanish in a neighborhood around that point. To this end, let $\varepsilon = \frac{|f(x)|}{2}$ which is positive as $f(x) \neq 0$. Then by the continuity of f, we know that there is some $\delta > 0$ such that $y \in N_\delta(x)$ implies $|f(y) - f(x)| < \varepsilon = \frac{|f(x)|}{2}$. But then $|f(y)| > \frac{|f(x)|}{2}$ and, in particular, $f(y) \neq 0$. Thus, $y \in Z(f)^c$ and $N_\delta(x) \subseteq Z(f)^c$. Hence, $Z(f)^c$ is open and $Z(f)$ is closed.

Solution: #4: To prove that $f(E)$ is dense in $f(X)$ we must prove that $f(X) \subseteq \overline{f(E)}$. To this end, take $y \in f(X)$ and we will show that $y \in \overline{f(E)}$. Write $y = f(x)$ for $x \in X$. Take some open set U such that $f(x) \in U$. Then $x \in f^{-1}(U)$ which is an open set since f is continuous. Since E is dense in X we know that $\overline{E} = X$. In particular, $x \in \overline{E}$ and thus either $x \in E$ or x is a limit point of E. If $x \in E$, then $y = f(x) \in f(E)$ and we are done since we have shown that $y \in \overline{f(E)}$.

If x is a limit point of E, since $x \in f^{-1}(U)$, there must exist a point $e \in E \cap f^{-1}(U)$ different from x. Then $f(e) \in f(E) \cap U$. If $f(e) = y$, then $y \in f(E)$ and we are done. Otherwise, we have found a point U different from y which is also in $f(E)$. Thus, y is a limit point of $f(E)$ and we are again done.

For the second part, consider the continuous function $h = f - g$. Then $h(p) = 0$ for all $p \in E$. By the first part, $h(X) \subseteq \overline{h(E)}$. But since $h(E) = \{0\}$, we have $h(X) \subseteq \overline{\{0\}} = \{0\}$. Thus $h(x) = 0$ for all $x \in X$ which implies $f(x) = g(x)$ for all $x \in X$.

Additional questions:

1. Define a function $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}.$$

Prove that f is not continuous at any point $p \in \mathbb{R}$.

Solution: Fix $p \in \mathbb{R}$ and assume that p is in $\mathbb{Q}$. Then fix $\varepsilon = 1/2$ and consider any $\delta > 0$. Since the irrationals are dense in $\mathbb{R}$, there is some irrational x satisfying $|x - p| < \delta$. Then $f(x) = 0$ while $f(p) = 1$. Thus $|f(x) - f(p)| = 1 > 1/2 = \varepsilon$. Hence, $f(x)$ is not continuous at p. Analogously, if p is not in $\mathbb{Q}$, then the same argument works as $\mathbb{Q}$ is also dense in $\mathbb{R}$.

2. Prove that $f : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ given by $f(x) = \sqrt{x}$ is continuous at all $p > 0$.
Hint: $(\sqrt{x} - \sqrt{p})(\sqrt{x} + \sqrt{p}) = x - p$.

Solution: Fix $\varepsilon > 0$ and set $\delta = \varepsilon/\sqrt{p}$. Then, for $|x - p| < \delta = \varepsilon \cdot \sqrt{p}$, we have

$$|f(x) - f(p)| = |\sqrt{x} - \sqrt{p}| = \frac{|x - p|}{\sqrt{x} + \sqrt{p}} < \frac{|x - p|}{\sqrt{p}} < \frac{\varepsilon\sqrt{p}}{\sqrt{p}} = \varepsilon.$$

Thus $f(x)$ is continuous at p.

3. Let $f : X \to Y$ be a continuous function between two metric spaces. For each of the following statements, either prove the statement or give some explicit counter-example.

(a) If $F \subseteq X$ is closed, then $f(F)$ is closed.

Solution: False. Take $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \arctan(x)$ and $F = \mathbb{R}$. Then F is closed, but $f(F) = (-\pi/2, \pi/2)$ is not closed. Alternatively, take $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x$ and take F equal to the graph of $y = 1/x$. Then F is closed, but $f(F) = \mathbb{R} - \{0\}$ which is not closed.

(b) If $U \subseteq X$ is open, then $f(U)$ is open.

Solution: False. Take $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ and $U = (-1, 1)$. Then U is open, but $f(U) = [0, 1)$ is not open.

(c) If $B \subseteq X$ is bounded, then $f(B)$ is bounded.

Solution: False. Take $f : (0, 1) \to R$ given by $f(x) = 1/x$ and $B = (0, 1)$. Then B is bounded, but $f(B) = (1, \infty)$ is unbounded.

4. Let $f : X \to Y$ be a continuous function between two metric spaces. For each of the following statements, either prove the statement or give some explicit counter-example.

(a) If $F \subseteq Y$ is closed, then $f^{-1}(F)$ is closed.

Solution: True. We have $f^{-1}(F)^c = f^{-1}(F^c)$ (check this!). Since F is closed, F^c is open, and since f is continuous $f^{-1}(F^c)$ is open. Thus $f^{-1}(F)^c$ is open which tells us that $f^{-1}(F)$ is closed.

(b) If $B \subseteq Y$ is bounded, then $f^{-1}(B)$ is bounded.

Solution: False. Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 0$ and take $B = \{0\}$. Then B is bounded, but $f^{-1}(B) = \mathbb{R}$ is unbounded.

(c) If $K \subseteq Y$ is compact, then $f^{-1}(K)$ is compact.

Solution: False. Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 0$ and take $K = \{0\}$. Then K is compact (being a single point), but $f^{-1}(K) = \mathbb{R}$ which is not compact as it is not bounded.

5. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Prove that $f(x)$ is continuous at $x = 0$.
(Hint: You may assume that $|\sin(y)| \leq 1$ for all $y \in \mathbb{R}$.)

Solution: Fix $\varepsilon > 0$ and set $\delta = \varepsilon$. Then for $0 < |x| < \delta = \varepsilon$, we have

$$|f(x) - f(0)| = |f(x)| = |x \sin(1/x)| = |x| \cdot |\sin(1/x)| \leq |x| < \varepsilon$$

and thus $f(x)$ is continuous at 0.