Can you imagine a mathematician writing Moby Dick? Let my name be Ishmael, let the captain's name be Ahab, let the boat's name be Pequod, and let the whale's name be as in the title. – Barry Cipra

Modern Algebra – Math 541 – Fall 2009 – R. Pollack
HW #2

1. Let $\mathrm{GL}_2(\mathbb{Z}_2)$ denote the collection of 2×2 matrices with entries in $\mathbb{Z}_2$ which have *non-zero* determinant. (We listed these matrices out in class.)
   (a) Make a multiplication table for $\mathrm{GL}_2(\mathbb{Z}_2)$.
   (b) Which pairs of matrices satisfy $a \cdot b = b \cdot a$?
   (c) Are there any elements which commute with *every* other matrix? That is, find all elements a in $\mathrm{GL}_2(\mathbb{Z}_2)$ such that $a \cdot b = b \cdot a$ for every b in $\mathrm{GL}_2(\mathbb{Z}_2)$.
   (d) For each matrix a, compute a, a^2, a^3, and so on until the pattern is clear. Determine the length of the repeating cycle for each matrix.

2. Consider the group D_3.
   (a) Which pairs of elements of D_3 satisfy $a \cdot b = b \cdot a$?
   (b) Are there any elements which commute with *every* other element? That is, find all elements a in D_3 such that $a \cdot b = b \cdot a$ for every b in D_3.
   (c) For each element a in D_3, compute a, a^2, a^3, and so on until the pattern is clear. Determine the length of the repeating cycle for each matrix.
   (d) Any observations? How does the group D_3 compare with the group $\mathrm{GL}_2(\mathbb{Z}_2)$?

3. Consider the group D_4. A multiplication table for this group is given in Chapter 1 of Gallian.
   (a) Which pairs of elements of D_4 satisfy $a \cdot b = b \cdot a$?
   (b) Are there any elements which commute with *every* other element? That is, find all elements a in D_4 such that $a \cdot b = b \cdot a$ for every b in D_4.
   (c) For each element a in D_4, compute a, a^2, a^3, and so on until the pattern is clear. Determine the length of the repeating cycle for each matrix.
   (d) Any observations?

4. Let $U(9)$ denote the collection of units in $\mathbb{Z}_9$ (see HW #1 question 2). There should be 6 such units.
   (a) Make a multiplication table for $U(9)$.
   (b) Which pairs of elements of $U(9)$ satisfy $a \cdot b = b \cdot a$?
   (c) For each element a in $U(9)$, compute a, a^2, a^3, and so on until the pattern is clear. Determine the length of the repeating cycle for each matrix.
   (d) Any observations? How does the group $U(9)$ compare with the two other groups on this problem set of size 6?

5. With all of these multiplication tables you have certainly by now observed (or used!) that the rows and columns in a multiplication table contain all distinct elements. Prove that this phenomenon is always true for the multiplication table of any group G.

   [Hint: Write out carefully in mathematical notation what it would mean for two elements of a column (or a row) of a multiplication table to be the same.]

6. Gallian question: Chapter 2 #17