Modern Algebra 2 – MA 542 – Spring 2019 – R. Pollack
HW #10

1. Find all automorphisms of $\mathbb{Q}(\sqrt[3]{2})$.

2. Find all non-zero homomorphisms from $\mathbb{Q}(\sqrt[3]{2})$ to $\overline{\mathbb{Q}}$. How does the number of maps you found compare to $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}]$?

3. Let $\zeta = e^{2\pi i/3}$ so that ζ has order 3 in $\mathbb{C}^\times$. Prove that the size of $\mathrm{Aut}(\mathbb{Q}(\sqrt[3]{2}, \zeta)/\mathbb{Q})$ is no more than 6.

4. Recall that the splitting field of $f(x) \in \mathbb{Q}[x]$ is simply the field $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ where $\alpha_1, \ldots, \alpha_n$ are the roots of $f(x)$.

   Let K denote the splitting field of $x^4 - 1$. Compute $[K : \mathbb{Q}]$. Prove that the size of $\mathrm{Aut}(K)$ is no more than $[K : \mathbb{Q}]$.

5. Let K denote the splitting field of $x^6 - 1$. Compute $[K : \mathbb{Q}]$. Prove that the size of $\mathrm{Aut}(K)$ is no more than $[K : \mathbb{Q}]$.

6. Let K denote the splitting field of $x^3 - 5$. Compute $[K : \mathbb{Q}]$. Prove that the size of $\mathrm{Aut}(K)$ is no more than $[K : \mathbb{Q}]$.

7. In the previous question, the size of $\mathrm{Aut}(K)$ is exactly equal to $[K : \mathbb{Q}]$. Determine up to isomorphism which group this is!

8. Find all of the subfields of $\mathbb{Q}(\sqrt[3]{2}, e^{2\pi i/3})$. (Hint: There are 6 in all counting $\mathbb{Q}$ and $\mathbb{Q}(\sqrt[3]{2}, e^{2\pi i/3})$.)

9. Find as many subfields of $\mathbb{Q}(i, \sqrt[4]{2})$ as you can! (Find at least 5.)

10. Is the field $\mathbb{Q}(i)$ a splitting field over $\mathbb{Q}$? Explain why or why not.

11. Is the field $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ a splitting field over $\mathbb{Q}$? Explain why or why not.

12. Is the field $\mathbb{Q}(\sqrt[3]{5})$ a splitting field over $\mathbb{Q}$? Explain why or why not.

13. Is the field $\mathbb{Q}(e^{2\pi i/11})$ a splitting field over $\mathbb{Q}$? Explain why or why not.

14. Is the field $\mathbb{Q}(\alpha)$ a splitting field over $\mathbb{Q}$ where α is the unique real root of $x^3 + x + 1$? Explain why or why not.