Modern Algebra 2 – MA 542 – Spring 2019 – R. Pollack
HW #10 solutions

1. Find all automorphisms of $\mathbb{Q}(\sqrt[3]{2})$.

   Solution: Any automorphisms of $\mathbb{Q}(\sqrt[3]{2})$ must send $\sqrt[3]{2}$ to another root of x^3-2, its minimal polynomial over $\mathbb{Q}$. But the other two roots of this polynomial are not in $\mathbb{R}$ and thus not in $\mathbb{Q}(\sqrt[3]{2})$. Hence the only automorphism of $\mathbb{Q}(\sqrt[3]{2})$ is the identity.

2. Find all non-zero homomorphisms from $\mathbb{Q}(\sqrt[3]{2})$ to $\overline{\mathbb{Q}}$. How does the number of maps you found compare to $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}]$?

   Solution: An embedding (i.e. a non-zero homomorphism) from $\mathbb{Q}(\sqrt[3]{2})$ to $\overline{\mathbb{Q}}$ must send $\sqrt[3]{2}$ to one of the three roots of $x^3 - 2$. Any of these three roots are possible and thus there are 3 such maps. Note also that $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$.

3. Let $\zeta = e^{2\pi i/3}$ so that ζ has order 3 in $\mathbb{C}^\times$. Prove that the size of $\mathrm{Aut}(\mathbb{Q}(\sqrt[3]{2}, \zeta)/\mathbb{Q})$ is no more than 6.

   Solution: If σ is such an automorphism then $\sigma(\sqrt[3]{2})$ must again be a root of $x^3 - 2$ and $\sigma(\zeta)$ must again be a root of $x^2 + x + 1$, its minimal polynomial. Thus, there are at most 3 choices for $\sigma(\sqrt[3]{2})$ and 2 choices for $\sigma(\zeta)$ making a total of 6 possible automorphisms as any automorphism is determined by its values on $\sqrt[3]{2}$ and on ζ.

4. Recall that the splitting field of $f(x) \in \mathbb{Q}[x]$ is simply the field $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ where $\alpha_1, \ldots, \alpha_n$ are the roots of $f(x)$.

   Let K denote the splitting field of $x^4 - 1$. Compute $[K : \mathbb{Q}]$. Prove that the size of $\mathrm{Aut}(K)$ is no more than $[K : \mathbb{Q}]$.

   Solution: The roots of $x^4 - 1$ are $\pm 1, \pm i$ and thus $K = \mathbb{Q}(\pm 1, \pm i) = \mathbb{Q}(i)$. Hence $[K : \mathbb{Q}] = 2$. Any automorphism of $\mathbb{Q}(i)$ is determined by its value on i and i must be mapped to i or $-i$ proving that there are at most 2 automorphisms.

5. Let K denote the splitting field of $x^6 - 1$. Compute $[K : \mathbb{Q}]$. Prove that the size of $\mathrm{Aut}(K)$ is no more than $[K : \mathbb{Q}]$.

   Solution: We have $x^6-1 = (x^3-1)(x^3+1) = (x-1)(x+1)(x^2+x+1)(x^2-x+1)$. Thus the roots of x^6-1 are $\pm 1, (-1 \pm \sqrt{-3})/2, (1 \pm \sqrt{-3})/2$. and thus $K = \mathbb{Q}(\pm 1, (-1 \pm \sqrt{-3})/2, (1 \pm \sqrt{-3})/2) = \mathbb{Q}(\sqrt{-3})$. Hence $[K : \mathbb{Q}] = 2$. Any automorphism of $\mathbb{Q}(\sqrt{-3})$ is determined by its value on $\sqrt{-3}$ and $\sqrt{-3}$ must be mapped to $\sqrt{-3}$ or $-\sqrt{-3}$ proving that there are at most 2 automorphisms.

6. Let K denote the splitting field of $x^3 - 5$. Compute $[K : \mathbb{Q}]$. Prove that the size of $\mathrm{Aut}(K)$ is no more than $[K : \mathbb{Q}]$.

   Solution: The roots of $x^3 - 5$ are $\sqrt[3]{5}, \zeta\sqrt[3]{5}, \zeta^2\sqrt[3]{5}$ where $\zeta = e^{2\pi/3}$ and thus $K = \mathbb{Q}(\sqrt[3]{5}, \zeta\sqrt[3]{5}, \zeta^2\sqrt[3]{5}) = \mathbb{Q}(\sqrt[3]{5}, \zeta)$. To determine $[K : \mathbb{Q}]$, note that $[K : \mathbb{Q}] = [K : \mathbb{Q}(\sqrt[3]{5})][\mathbb{Q}(\sqrt[3]{5}) : \mathbb{Q}]$. Computing $[\mathbb{Q}(\sqrt[3]{5}) : \mathbb{Q}$ is easy as $x^3 - 5$ is irreducible over $\mathbb{Q}$ by the Eisenstein criteria with $p = 5$. To compute $[K : \mathbb{Q}(\sqrt[3]{5})]$ we note that $x^2 + x + 1$ is the minimal polynomial of ζ over $\mathbb{Q}(\sqrt[3]{5})$. Indeed, this polynomial has no roots in $\mathbb{Q}(\sqrt[3]{5})$ as both roots are not in $\mathbb{R}$ and $\mathbb{Q}(\sqrt[3]{5}) \subseteq \mathbb{R}$. Thus $[K : \mathbb{Q}(\sqrt[3]{5})] = 2$ and $[K : \mathbb{Q}] = 6$. Lastly, any automorphism of K is determined by its value on $\sqrt[3]{5}$ and on ζ. We have that $\sqrt[3]{5}$ has at most 3 possible images and ζ has at most 2 possible imagines. Thus the size of this automorphism group is at most 6.

7. In the previous question, the size of $\text{Aut}(K)$ is exactly equal to $[K : \mathbb{Q}]$. Determine up to isomorphism which group this is!

   Solution: We have that $\text{Aut}(K)$ is a group of size 6. We will check that this group is non-abelian which forces it to be isomophic to S_3, the only non-abelian group of size 6. To this end, let σ be defined by $\sigma(\sqrt[3]{5}) = \zeta\sqrt[3]{5}$ and $\sigma(\zeta) = \zeta$ and let τ be defined by $\tau(\sqrt[3]{5}) = \sqrt[3]{5}$ and $\tau(\zeta) = \zeta^2$. Then

   $$\sigma(\tau(\sqrt[3]{5})) = \sigma(\sqrt[3]{5}) = \zeta\sqrt[3]{5}$$

   while

   $$\tau(\sigma(\sqrt[3]{5})) = \tau(\zeta\sqrt[3]{5}) = \tau(\zeta)\tau(\sqrt[3]{5}) = \zeta^2\zeta\sqrt[3]{5} = \sqrt[3]{5}.$$

8. Find all of the subfields of $\mathbb{Q}(\sqrt[3]{2}, e^{2\pi i/3})$. (Hint: There are 6 in all counting $\mathbb{Q}$ and $\mathbb{Q}(\sqrt[3]{2}, e^{2\pi i/3})$.)

   Solution: The subfields of $\mathbb{Q}(\sqrt[3]{2}, e^{2\pi i/3})$ are $\mathbb{Q}$, $\mathbb{Q}(\sqrt[3]{2})$, $\mathbb{Q}(e^{2\pi/3})$, $\mathbb{Q}(\sqrt[3]{2}e^{2\pi/3})$, $\mathbb{Q}(\sqrt[3]{2}e^{4\pi/3})$ and $\mathbb{Q}(\sqrt[3]{2}, e^{2\pi i/3})$.

9. Find as many subfields of $\mathbb{Q}(i, \sqrt[4]{2})$ as you can! (Find at least 5.)

   Solution: The "easy to find" subfields of $\mathbb{Q}(i, \sqrt[4]{2})$ are: $\mathbb{Q}$, $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt[4]{2})$, $\mathbb{Q}(\sqrt[4]{2}i)$ and $\mathbb{Q}(i, \sqrt[4]{2})$. There is also $\mathbb{Q}(\sqrt{2})$ as $\sqrt{2} = \sqrt[4]{2}^2$. And thus we also have $\mathbb{Q}(\sqrt{2}i)$ as a subfield as well as $\mathbb{Q}(i, \sqrt{2})$. There are two more as well, but they are a little harder to find...

10. Is the field $\mathbb{Q}(i)$ a splitting field over $\mathbb{Q}$? Explain why or why not.

    Solution: Yes! It is the splitting field of $x^2 + 1$.

11. Is the field $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ a splitting field over $\mathbb{Q}$? Explain why or why not.

    Solution: Yes! It is the splitting field of $(x^2 - 2)(x^2 - 3)$.

12. Is the field $\mathbb{Q}(\sqrt[3]{5})$ a splitting field over $\mathbb{Q}$? Explain why or why not.

    Solution: No. The polynomial $x^3 - 5$ has only one root in this field as the other roots are not in $\mathbb{R}$ and $\mathbb{Q}(\sqrt[3]{5})$ is contained in $\mathbb{R}$. Hence the field is not normal and thus cannot be a splitting field. (This follows from the TFAE theorem in class with 3 parts.)

13. Is the field $\mathbb{Q}(e^{2\pi i/11})$ a splitting field over $\mathbb{Q}$? Explain why or why not.

    Solution: Yes! It is the splitting field of $x^{11} - 1$. Indeed, the roots of this polynomial are all of the form $e^{2k\pi i/11}$ for $k = 1, \ldots, 11$ and thus they are all powers of $e^{2\pi i/11}$.

14. Is the field $\mathbb{Q}(\alpha)$ a splitting field over $\mathbb{Q}$ where α is the unique real root of $x^3 + x + 1$? Explain why or why not.

    Solution: No. The polynomial $x^3 + x + 1$ has only 1 root in $\mathbb{Q}(\alpha)$ as $\mathbb{Q}(\alpha)$ is contained in $\mathbb{R}$ and the other 2 roots are not in $\mathbb{R}$. Thus $\mathbb{Q}(\alpha)$ is not a normal extension of $\mathbb{Q}$ and hence not a splitting field over $\mathbb{Q}$ (as in question 12).