Introduction to Analysis – MA 542 – Fall 2019 – R. Pollack
HW #4 Solutions

Section 26:

1) Let $\varphi : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$ be a ring homomorphism. Let $e_1 = (1,0)$ and $e_2 = (0,1)$. Then the values of $\varphi(e_1)$ and $\varphi(e_2)$ completely determine φ as

$$\varphi((a,b)) = a\varphi(e_1) + b\varphi(e_2)$$

since φ is an additive homomorphism.

Write $\varphi(e_1) = f_1$ and $\varphi(e_2) = f_2$. Since $e_1 \cdot e_1 = e_1$ and φ is a ring homomorphism, we must have that $\varphi(e_1)\cdot\varphi(e_1) = \varphi(e_1)$ and thus $f_1 \cdot f_1 = f_1$. Likewise, $e_2 \cdot e_2 = e_2$ gives that $f_2 \cdot f_2 = f_2$. Further, $e_1 \cdot e_2 = (0,0)$ implies $f_1 \cdot f_2 = (0,0)$.

We claim now that any choice of f_1 and f_2 such that $f_1^2 = f_1$, $f_2^2 = f_2$ and $f_1 \cdot f_2 = (0,0)$ gives rise to a ring homomorphism where $\varphi(e_1) = f_1$ and $\varphi(e_2) = f_2$. Indeed, the corresponding φ is clearly an additive ring homomorphism. To check that it is a multiplicative homomorphism, note that

$$\varphi((a,b)\cdot(c,d)) = \varphi((ac,bd)) = \varphi(ace_1 + bde_2) = acf_1 + bdf_2$$

while

$$\begin{aligned}\varphi((a,b))\cdot\varphi((c,d)) &= \varphi(ae_1 + be_2)\cdot\varphi(ce_1 + de_2)\\ &= (af_1 + bf_2)\cdot(cf_1 + df_2)\\ &= acf_1f_1 + adf_1f_2 + bcf_2f_1 + bdf_2f_2\\ &= acf_1 + bdf_2\end{aligned}$$

since $f_1^2 = f_1$, $f_2^2 = f_2$ and $f_1f_2 = 0$. Thus $\varphi((a,b)\cdot(c,d)) = \varphi((a,b))\cdot\varphi((c,d))$ and φ is a ring homomorphism.

Finally, we must compute all possible values of $f_1 = (a,b)$ and $f_2 = (c,d)$ satisfying $f_1^2 = f_1$, $f_2^2 = f_2$ and $f_1f_2 = 0$. Since $f_1^2 = f_1$, we have $(a,b)\cdot(a,b) = (a,b)$ which implies that $a^2 = a$ and $b^2 = b$. Since $a,b \in \mathbb{Z}$, we must have that $a = 0$ or 1 and $b = 0$ or 1. Similarly, by considering $f_2 \cdot f_2 = f_2$ we deduce that $c = 0$ or 1 and $d = 0$ or 1. Further, $f_1f_2 = (0,0)$ implies that $(ac,bd) = (0,0)$ and thus $ac = 0$ and $bd = 0$.

There are nine such possible values of a,b,c,d solving these equations. Namely,

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix} 0 & 0\\ 1 & 0\end{pmatrix}, \begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix} 1 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}.$$

3) We first consider the principal ideals $I = x\mathbb{Z}_{12}$. We have

$$\begin{aligned}
x = 0 &: I = 0\mathbb{Z}_{12} = \{0\}\\
x = 1 &: I = x\mathbb{Z}_{12} = \mathbb{Z}_{12}\\
x = 2 &: I = x\mathbb{Z}_{12} = \{0,2,4,6,8,10\}\\
x = 3 &: I = x\mathbb{Z}_{12} = \{0,3,6,9\}\\
x = 4 &: I = x\mathbb{Z}_{12} = \{0,4,8\}\\
x = 5 &: I = x\mathbb{Z}_{12} = \{0,5,10,3,8,1,6,11,4,9,2,7\} = \mathbb{Z}_{12}\\
x = 6 &: I = x\mathbb{Z}_{12} = \{0,6\}\\
x = 7 &: I = x\mathbb{Z}_{12} = \{0,7,2,9,4,11,6,1,8,3,10,5\} = \mathbb{Z}_{12}\\
x = 8 &: I = x\mathbb{Z}_{12} = \{0,8,4\} = 4\mathbb{Z}_{12}\\
x = 9 &: I = x\mathbb{Z}_{12} = \{0,9,6,3\} = 3\mathbb{Z}_{12}\\
x = 10 &: I = x\mathbb{Z}_{12} = \{0,10,8,6,4,2\} = 2\mathbb{Z}_{12}\\
x = 11 &: I = x\mathbb{Z}_{12} = \{0,11,10,9,8,7,6,5,4,3,2,1\} = \mathbb{Z}_{12}
\end{aligned}$$

This gives 6 distinct principal ideals: $\langle 0\rangle, \langle 1\rangle, \langle 2\rangle, \langle 3\rangle, \langle 4\rangle, \langle 6\rangle$. It turns out that all ideals in this ring are principal and so there are no additional ideals to consider.

Now for the factors rings. In each case $\mathbb{Z}_{12}/\langle n\rangle$ is isomorphic to $\mathbb{Z}_{\gcd(n,12)}$ if we interpret $\mathbb{Z}_1$ to be the zero ring.

4)

Addition table:

$+$	$0+8\mathbb{Z}$	$2+8\mathbb{Z}$	$4+8\mathbb{Z}$	$6+8\mathbb{Z}$
$0+8\mathbb{Z}$	$0+8\mathbb{Z}$	$2+8\mathbb{Z}$	$4+8\mathbb{Z}$	$6+8\mathbb{Z}$
$2+8\mathbb{Z}$	$2+8\mathbb{Z}$	$4+8\mathbb{Z}$	$6+8\mathbb{Z}$	$0+8\mathbb{Z}$
$4+8\mathbb{Z}$	$4+8\mathbb{Z}$	$6+8\mathbb{Z}$	$0+8\mathbb{Z}$	$2+8\mathbb{Z}$
$6+8\mathbb{Z}$	$6+8\mathbb{Z}$	$0+8\mathbb{Z}$	$2+8\mathbb{Z}$	$4+8\mathbb{Z}$

Multiplication table:

$\cdot$	$0+8\mathbb{Z}$	$2+8\mathbb{Z}$	$4+8\mathbb{Z}$	$6+8\mathbb{Z}$
$0+8\mathbb{Z}$	$0+8\mathbb{Z}$	$0+8\mathbb{Z}$	$0+8\mathbb{Z}$	$0+8\mathbb{Z}$
$2+8\mathbb{Z}$	$0+8\mathbb{Z}$	$4+8\mathbb{Z}$	$0+8\mathbb{Z}$	$4+8\mathbb{Z}$
$4+8\mathbb{Z}$	$0+8\mathbb{Z}$	$0+8\mathbb{Z}$	$0+8\mathbb{Z}$	$0+8\mathbb{Z}$
$6+8\mathbb{Z}$	$0+8\mathbb{Z}$	$4+8\mathbb{Z}$	$0+8\mathbb{Z}$	$4+8\mathbb{Z}$

From these tables we can see that $2\mathbb{Z}/8\mathbb{Z}$ is not isomorphic to $\mathbb{Z}_4$. Indeed, there is no multiplicative identity in $2\mathbb{Z}/8\mathbb{Z}$ while there is one in $\mathbb{Z}_4$.

10)

a. Yes...I guess so.

b. False. For instance $\mathbb{Z} \to \mathbb{Z}[i]$ by $a \mapsto a + 0i$ is a ring homomorphism, but $\mathbb{Z}$ is not an ideal of $\mathbb{Z}[i]$.

c. True.

d. False. If an ideal contains 1, then it must be the whole ring.

e. True.

f. False. As above with $\mathbb{Z}$ and $\mathbb{Z}[i]$.

g. True

h. True.

i. True.

j. Yes.

12) For any prime p, $\mathbb{Z}_p$ is a field and $\mathbb{Z}_p$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

13) We have that $\mathbb{Z}$ is an integral domain, but $\mathbb{Z}/4\mathbb{Z}$ is not an integral domain as $2 \cdot 2 = 0$.

14) We have that $\mathbb{Z}_6$ is not an integral domain as $2 \cdot 3 = 0$, but $\mathbb{Z}_6/2\mathbb{Z}_6$ is isomorphic to $\mathbb{Z}_3$ which is a field and thus an integral domain.

16)

a. This is a silly question. I think the answer the textbook is looking for is that one shouldn't write $r, s \in R/N$; one should instead write $r+N, s+N \in R/N$. But maybe r is the notation for a coset of the form $a+N$...

b. see above

c. We have R/N is commutative iff $(r+N)(s+N) = (s+N)(r+N)$ for all $r, s \in R$ iff $rs+N = sr+N$ iff $rs - sr \in N$.

17) Write $\mathbb{Z}[\sqrt{2}]$ for R in this question. Then $\mathbb{Z}[\sqrt{2}]$ is closed under $+$ and $\cdot$ as

$$a + b\sqrt{2} + c + d\sqrt{2} = a + c + (b+d)\sqrt{2}$$

and

$$(a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2}.$$

Clearly $0 \in \mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{2}]$ has additive inverses as $-(a+b\sqrt{2}) = -a - b\sqrt{2}$. The remaining ring axioms all follow as $\mathbb{Z}[\sqrt{2}]$ is contained in $\mathbb{R}$.

For R', we have

$$\begin{pmatrix} a & 2b \\ b & a \end{pmatrix} + \begin{pmatrix} c & 2d \\ d & c \end{pmatrix} = \begin{pmatrix} a+c & 2(b+d) \\ b+d & a+c \end{pmatrix}$$

and

$$\begin{pmatrix} a & 2b \\ b & a \end{pmatrix} \cdot \begin{pmatrix} c & 2d \\ d & c \end{pmatrix} = \begin{pmatrix} ac+2bd & 2(ad+bc) \\ ad+bc & ac+2bd \end{pmatrix}$$

and thus addition and multiplication are closed on R'. Again clearly $0 \in R'$ and $-\begin{pmatrix} a & 2b \\ b & a \end{pmatrix} = \begin{pmatrix} -a & 2(-b) \\ -b & -a \end{pmatrix}$ making R' a subring.

Finally, consider the map $\varphi : \mathbb{Z}[\sqrt{2}] \to R'$ by $\varphi(a + b\sqrt{2}) = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$. We have

$$\varphi((a + b\sqrt{2})(c + d\sqrt{2})) = \varphi((ac + 2bd) + (ad + bc)\sqrt{2}) = \begin{pmatrix} ac+2bd & 2(ad+bc) \\ ad+bc & ac+2bd \end{pmatrix}$$

while

$$\varphi(a + b\sqrt{2})\varphi(c + d\sqrt{2}) = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} \cdot \begin{pmatrix} c & 2d \\ d & c \end{pmatrix} = \begin{pmatrix} ac+2bd & 2(ad+bc) \\ ad+bc & ac+2bd \end{pmatrix}.$$

Seeing that φ is an additive homomorphism is easy and thus φ is a ring homomorphism. Clearly φ is surjective. We then just need to compute its kernel. We have $a + b\sqrt{2} \in \ker(\varphi)$ implies $\begin{pmatrix} a & 2b \\ b & a \end{pmatrix} = 0$ which implies that $a = b = 0$. Thus $\ker(\varphi) = \{0\}$ and φ is injective. Hence, φ is an isomorphism.

24) The only ideals of a field F are $\{0\}$ and F as proven in class. Thus, the only possible quotients are F/F and $F/\{0\}$. Here F/F is the zero ring while $F/\{0\}$ is isomorphic to F.

26) Clearly $0 \in I_a$. For $x, y \in I_a$ we have $ax = 0$ and $ay = 0$. Thus $0 = ax + ay = a(x+y)$ which implies $a(x+y) = 0$. $x + y \in I_a$. Lastly, for $x \in I_a$ and $r \in R$, we have $ax = 0$ and thus $axr = 0$ which implies $xr \in I_a$. Hence, I_a is an ideal of R.

27) I'll check this for 2 ideals, but the same proof works for arbitrarily many ideals. Let I and J be ideals of R. First note that $0 \in I \cap J$. Next, for $x, y \in I \cap J$, we have $x, y \in I$ and $x, y \in J$. Since I and J are ideals, we have $x + y \in I$ and $x + y \in J$. Hence, $x + y \in I \cap J$ as desired. Lastly, take $x \in I \cap J$ and $r \in R$. Then since $x \in I$ and $x \in J$, we have $rx, xr \in I$ and $rx, xr \in J$ again since I and J are ideals. Thus, $rx, xr \in I \cap J$ and $I \cap J$ is an ideal of R.

30) Let I be the nilradical of R. Clearly $0 \in I$ since $0^1 = 0$. Now take $x, y \in I$ and we need to check that $x + y \in I$. Let n, m be integers such that $x^n = y^m = 0$. Then

$$(x+y)^{n+m} = \sum_{j=0}^{n+m} \binom{n+m}{j} x^j y^{m+n-j}.$$

In this sum, if $j \geq n$, then $x^j = 0$ and thus $x^j y^{m+n-j} = 0$. But if $j < n$, then $m + n - j \geq m$ and hence $y^{m+n-j} = 0$. Hence every term in the sum vanishes and $(x+y)^{n+m} = 0$. Thus $x + y \in I$ as desired.

Finally, for $r \in R$ and $x \in I$, we need to check that $rx \in I$. But this is easy as $x \in I$ implies that there is some n such that $x^n = 0$. But then $(rx)^n = r^n x^n = 0$ and so $rx \in I$.

31) For $\mathbb{Z}_{12}$, we have $x^n = 0 \in \mathbb{Z}_{12}$ implies $4|x^n$ and $3|x^n$. In particular, $2|x$ and $3|x$ which implies $6|x$. Since $6^2 = 0$ in $\mathbb{Z}_{12}$, we have that 6 is in fact nilpotent in $\mathbb{Z}_{12}$ and thus, the nilradical of $\mathbb{Z}_{12}$ is the principal ideal generated by 6.

For $\mathbb{Z}$, since $\mathbb{Z}$ is an integral domain, the only nilpotent element is 0 and the nilradical is just $\{0\}$.

For $\mathbb{Z}_{32}$, we have $x^n = 0 \in \mathbb{Z}_{32}$ implies $2^5|x^n$ which implies that $2|x$. Since $2^5 = 0$ in $\mathbb{Z}_{32}$, we have that 2 is in fact nilpotent in $\mathbb{Z}_{32}$ and thus, the nilradical of $\mathbb{Z}_{32}$ is the principal ideal generated by 2.

32) Assume $x + N$ is nilpotent in R/N. Then $(x + N)^n = 0$ for some n and thus $x^n + N = 0$. This implies that $x^n \in N$. Thus x^n is nilpotent in R and there exists m such that $(x^n)^m = 0$ in R. But then $x^{nm} = 0$ in R and x is nilpotent in R. Hence $x \in N$ and $x + N = 0$ in R/N as desired.