Introduction to Analysis – MA 542 – Fall 2019 – R. Pollack
HW #5 Solutions

Section 27:

5) This quotient is a field iff $x^2 + c$ is an irreducible polynomial in $\mathbb{Z}_3[x]$ and this occurs iff $x^2 + c$ has no roots in $\mathbb{Z}_3$. Thus, $c = 1$ is the only possibility as $x^2$ and $x^2 + 2$ both have roots.

6) This quotient is a field iff $x^3 + x^2 + c$ is an irreducible polynomial in $\mathbb{Z}_3[x]$ and this occurs iff $x^3 + x^2 + c$ has no roots in $\mathbb{Z}_3$. Thus, $c = 2$ is the only possibility as $x^3 + x^2$ and $x^3 + x^2 + 1$ both have roots.

8) This quotient is a field iff $x^2 + x + c$ is an irreducible polynomial in $\mathbb{Z}_3[x]$ and this occurs iff $x^2 + x + c$ has no roots in $\mathbb{Z}_3$. Thus, $c = 1, 2$ are the only possibilities as $x^2 + x, x^2 + x + 3, x^2 + x + 4$ and $x^3 + x^2 + 1$ all have roots.

14)

a. False. $\langle 0 \rangle$ is a prime ideal of $\mathbb{Z}$ but is not maximal.

b. True.

c. True.

d. False. The prime field here is $\mathbb{Q}$.

e. True.

f. True. For instance, $\mathbb{Z}_2[x]/\langle x \rangle$.

g. True.

h. False. $\langle x^2 \rangle$ is not a prime ideal.

i. True.

j. False. $\langle 0 \rangle$ is not a maximal ideal.

15) $I = \mathbb{Z} \times 2\mathbb{Z}$ is maximal as $(\mathbb{Z} \times \mathbb{Z})/(\mathbb{Z} \times 2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ which is a field. (To see this isomorphism, consider the map $\varphi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ given by $\varphi((x, y)) = y + 2\mathbb{Z}$. Then $\ker(\varphi) = I$ and apply the first isomorphism theorem.)

16) $I = \mathbb{Z} \times \{0\}$ is prime as $(\mathbb{Z} \times \mathbb{Z})/(\mathbb{Z} \times \{0\}) \cong \mathbb{Z}$ and $\mathbb{Z}$ is an integral domain. However, $I$ is not maximal as it is contained in the example from #15.

17) $I = 6\mathbb{Z} \times \{0\}$ is not prime as $(2, 0) \cdot (3, 0) \in I$ but neither of these elements is in $I$.

18) Since $x^2-5x+6 = (x-2)(x-3)$, we have $\langle x^2-5x+6 \rangle$ is NOT a maximal ideal and hence $\mathbb{Q}[x]/\langle x^2-5x+6 \rangle$ is not a field. We can see this directly as well as $(x - 2 + \langle x^2 - 5x + 6 \rangle)(x - 3 + \langle x^2 - 5x + 6 \rangle) = 0$ in $\mathbb{Q}[x]/\langle x^2 - 5x + 6 \rangle$ and thus $\mathbb{Q}[x]/\langle x^2 - 5x + 6 \rangle$ is not even an integral domain.

24) Let $I$ be a prime ideal. Then $R/I$ is a finite integral domain. Since finite integral domains are fields (proven in class), we have that $R/I$ is a field and hence $I$ is a maximal ideal.

30) Let $I$ be a non-trivial proper prime ideal of $F[x]$. This means that $I \neq 0$ and $I \neq F[x]$ and that $I$ is prime. Since all ideals of $F[x]$ are principal, we have that $I = \langle \pi \rangle$ for some $\pi \in F[x]$. To prove $I$ is maximal, let $A = \langle g \rangle$ be some ideal containing $I$. Then $\pi \in A$ implies that $\pi = g \cdot g'$. Thus $g \cdot g' \in I$ and since $I$ is prime we have $g \in I$ or $g' \in I$. In the first case, we then have that $A = I$ while in the second case, $g' = \pi h$ so that $\pi = gg' = g\pi h$. Cancelling gives $gh = 1$ and thus $g$ and $h$ are constants. This implies that $A = R$ and that $I$ is a maximal ideal.

32) We first check that $N$ is an ideal. Clearly $0 \in N$. Let $r_1 f + s_1 g, r_2 f + s_2 g$ be in $N$. Then

$$r_1 f + s_1 g + r_2 f + s_2 g = (r_1 + r_2) f + (s_1 + s_2) g$$

is again in $N$. Further, for $h \in F[x]$ and $r_1 f + s_1 g \in N$, we have

$$h(r_1 f + s_1 g) = h r_1 f + h s_1 g$$

is again in $N$. Thus $N$ is an ideal.

For the second question, assume that both $f(x)$ and $g(x)$ are irreducible. Then $\langle f \rangle$ and $\langle g \rangle$ are both maximal ideals. Since $N \supseteq \langle f \rangle$, we then have that $N = F[x]$ or $N = \langle f \rangle$. But the question poses that $N \neq F[x]$ and thus $N = \langle f \rangle$. The analogous argument shows that $N = \langle g \rangle$. But then $\langle f \rangle = \langle g \rangle$ which implies that $f$ and $g$ have the degree.

34) (a) Clearly $0 \in A + B$. Take $a_1 + b_1$ and $a_2 + b_2$ in $A + B$. Then

$$a_1 + b_1 + a_2 + b_2 = (a_1 + a_2) + (b_1 + b_2) \in A + B.$$

For $r \in R$ and $a + b \in A + B$, we have

$$r \cdot (a + b) = ra + rb \in A + B$$

since $A$ and $B$ are ideals.
(b) It's clear that $A \subseteq A + B$ as $a = a + 0$ and $0 \in B$. Likewise for $B \subseteq A + B$.

35) (a) Clearly the sum of two elements of $AB$ is again in $AB$ as a sum of $n$ terms of the form $ab$ with a sum of $m$ terms of this form is simply a sum of $m + n$ terms of this form. For scalars, let $r \in R$ and $\sum_{i=1}^{n} a_i b_i \in AB$. Then

$$r \cdot \sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} r a_i b_i$$

which is in $AB$ as $A$ is an ideal.
b) Any element $\sum_{i=1}^{n} a_i b_i$ of $AB$ is in $A$ as $A$ is an ideal and $a_i \in A$. Likewise, this sum is in $B$ as $b_i \in B$ and $B$ is an ideal. Thus, $AB$ is contained in $A \cap B$.