Introduction to Analysis – MA 542 – Fall 2019 – R. Pollack
HW #8 Solutions

Section 31:
20) Powers of an element α in a finite extension E/F eventually satisfy a linear relation over F which is thus a polynomial relation on α.

21) If $\{\alpha_i\}$ is a basis of K/E and $\{\beta_i\}$ is a basis of E/F, then $\{\alpha_i\beta_j\}$ is a basis of K/F. Indeed, one can write any element in K as a linear combination of the α_i with coefficients in E and those coefficients one can write as a linear combination of the β_i with coefficients in F which yields a linear combination of the $\alpha_i\beta_j$ with coefficients in F.

24) If $x^2 - 3$ had a root in $\mathbb{Q}(\sqrt[3]{2})$ then $\mathbb{Q}(\sqrt{3})$ would be a subfield of $\mathbb{Q}(\sqrt[3]{2})$. But then $[\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 2$ would divide $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ which is impossible.

28) Clearly $\mathbb{Q}(\sqrt{a}+\sqrt{b}) \subseteq \mathbb{Q}(\sqrt{a},\sqrt{b})$ as $\sqrt{a}+\sqrt{b} \in \mathbb{Q}(\sqrt{a},\sqrt{b})$. To see the reverse inclusion, note that

$$\frac{a-b}{\sqrt{a}+\sqrt{b}} = \sqrt{a}-\sqrt{b}.$$

Since $a-b$ and $\sqrt{a}+\sqrt{b}$ are both in $\mathbb{Q}(\sqrt{a}+\sqrt{b})$, their ratio is in $\mathbb{Q}(\sqrt{a}+\sqrt{b})$ and thus $\sqrt{a}-\sqrt{b} \in \mathbb{Q}(\sqrt{a}+\sqrt{b})$. Further, we have

$$\frac{(\sqrt{a}+\sqrt{b})+(\sqrt{a}-\sqrt{b})}{2} = \sqrt{a}$$

and

$$\frac{(\sqrt{a}+\sqrt{b})-(\sqrt{a}-\sqrt{b})}{2} = \sqrt{b}.$$

Since both $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$ are in $\mathbb{Q}(\sqrt{a}+\sqrt{b})$, we then deduce that $\sqrt{a}$ and $\sqrt{b}$ are in $\mathbb{Q}(\sqrt{a}+\sqrt{b})$. Hence $\mathbb{Q}(\sqrt{a},\sqrt{b}) \subseteq \mathbb{Q}(\sqrt{a}+\sqrt{b})$ and thus these two fields are the same.

31) It is clear that if K/F is algebraic then both K/E and E/F are algebraic. Indeed, for any α in K, α satisfies a non-zero polynomial with coefficients in F and so it certainly satisfies one with coefficients in the larger field E. This shows that K/E is algebraic. To see that E/F is algebraic, take any $\alpha \in E$. But then α is also in K and thus is algebraic over F. Hence E/F is algebraic.

To see the converse, assume that both K/E and E/F are algebraic and take $\alpha \in K$. Then α satisfies a non-zero polynomial with coefficients in E, say $x^n + a_{n-1}x^{n-1} + a_1x + a_0$. Consider the field $F(a_0, a_1, \ldots, a_{n-1})$. Since each $a_i \in E$ and E/F is algebraic, we deduce that $F(a_0, a_1, \ldots, a_{n-1})$ is finite over F. (We proved this in class or see Theorem 31.11.). Also, α is algebraic over $F(a_0, a_1, \ldots, a_{n-1})$ since its minimal polynomial (by construction) has coefficients in this field. Thus, $F(a_0, a_1, \ldots, a_{n-1}, \alpha)$ is finite over $F(a_0, a_1, \ldots, a_{n-1})$. Since $F(a_0, a_1, \ldots, a_{n-1}, \alpha)$ is finite over $F(a_0, a_1, \ldots, a_{n-1})$ and $F(a_0, a_1, \ldots, a_{n-1})$ is finite over F, we deduce that $F(a_0, a_1, \ldots, a_{n-1}, \alpha)$ is finite over F (as degrees are multiplicative). But finite extensions are algebraic and thus α is algebraic over F as desired.

33) Take a non-constant polynomial $f(x)$ in $F_E[x]$. Write $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ with each $a_i \in F_E$ so that $a_i \in E$ and a_i is algebraic over F. As E is algebraically closed, there is some $\alpha \in E$ which is a zero of $f(x)$. But then $F_E(\alpha)/F_E$ is algebraic. Since F_E/F is algebraic by definition, exercise #31 implies that $F_E(\alpha)/F$ is algebraic. Thus, α is algebraic over F and then by definition $\alpha \in F_E$. Hence F_E is algebraically closed.

36) We have that $2^{1/n}$ is in $\overline{\mathbb{Q}}$ for all n. Since $x^n - 2$ is the minimal polynomial of $2^{1/n}$ over $\mathbb{Q}$ (it is irreducible by Eisenstein $p = 2$), we have $[\mathbb{Q}(2^{1/n}) : \mathbb{Q}] = n$. But then $[\overline{\mathbb{Q}} : \mathbb{Q}] \geq n$ for all n and hence $\overline{\mathbb{Q}}/\mathbb{Q}$ is an infinite extension.