Topology – MA 564 – Spring 2015 – R. Pollack
HW #1 Solutions

Complete each of the following exercises.

Gamelin & Greene
Chapter 1: 1,2,4,5

#1) See solutions in G & G for (a) and the solution of (b) is similar. For (c), this is false. Consider $U = \mathbb{R}$, $V = \mathbb{R}$ and $W = \mathbb{R}$. Then

$$U\backslash(V\backslash W) = \mathbb{R}\backslash(\mathbb{R}\backslash\mathbb{R}) = \mathbb{R}\backslash\emptyset = \mathbb{R},$$

while

$$(U\backslash V)\backslash W = (\mathbb{R}\backslash\mathbb{R})\backslash\mathbb{R} = \emptyset\backslash\mathbb{R} = \emptyset.$$

For (d), again see G & G.

#2) For the first part we need to check that the discrete metric satisfies the three axioms of a metric space. For the first we need to check that $d(x, y) = 0$ iff $x = y$. But this is immediate from the definition of the discrete metric (since $x \neq y$ iff $d(x, y) = 1$). For the second, we need to see $d(x, y) = d(y, x)$ but the definition of the discrete metric is symmetric in x and y and so this is immediate. For the last, we need $d(x, z) \leq d(x, y) + d(y, z)$. Since the values of d are only 0 and 1, the only way this inequality could fail is if the left hand side was 1 but the right hand side was 0. But if the right hand side was 0, then $d(x, y) = d(y, z) = 0$ which implies $x = y = z$. But then $d(x, z) = 0$ and the left hand side is also 0.

#4) We need to show that $(0, 1]$ is neither open or closed. To see that $(0, 1]$ is not open, we show that $1 \in (0, 1]$ is not an interior point. To see this, take any $r > 0$ and we note that $B_r(1)$ is not completely contained within $(0, 1]$ — for instance, $B_r(1)$ contains $1 + \frac{r}{2}$ but $(0, 1]$ does not. To see that $(0, 1]$ is not closed, we show that 0 is an adherent point (but is not in (0,1]). To see this, take any $r > 0$ and we need to see that $B_r(0)$ is not disjoint from $(0, 1]$. But this is clear as both sets contain $\frac{r}{2}$.

#5) We must show that $B(S)$, the set of bounded functions on S, is a metric space under the metric defined in (1.6). For $f, g \in B(S)$, we need $d(f, g) = 0$ iff $f = g$. Clearly, $d(f, f) = 0$. Conversely, if $d(f, g) = 0$ then $\sup_{s\in S} |f(s) - g(s)| = 0$. In particular, $|f(s) - g(s)| = 0$ for all $s \in S$ since if $|f(s) - g(s)| > 0$ for some s, then the supremum would also be greater than 0. Hence $f(s) = g(s)$ for all $s \in S$ which means exactly that $f = g$ in $B(S)$. For the symmetry property, this is clear as $|f(s) - g(s)| = |g(s) - f(s)|$. Lastly, for the triangle inequality, for $f, g, h \in B(S)$, we have

$$\begin{aligned} d(f, g) &= \sup_{s\in S} |f(s) - g(s)| \\ &= \sup_{s\in S} |f(s) - h(s) + h(s) - g(s)| \\ &\leq \sup_{s\in S} |f(s) - h(s)| + |h(s) - g(s)| \\ &= \sup_{s\in S} |f(s) - h(s)| + \sup_{s\in S} |h(s) - g(s)| \\ &= d(f, h) + d(h, g). \end{aligned}$$

Here the third line is derived from the triangle inequality for regular absolute value and the fourth line is a standard property of supremum.

Freiwald
Chapter 2, pg. 74–75: E1(a), E2, E8(a,c)

E1(a): This is false! A counter-example comes from Additional Question #2. Indeed, there I show that in the 3-adic metric on $\mathbb{Z}$, we have $B_1(1) = \{3n + 1 \mid n \in \mathbb{Z}\}$ while $B_1(4) = \{3n + 4 \mid n \in \mathbb{Z}\}$. But

$\{3n+1 \mid n \in \mathbb{Z}\} = \{3n+4 \mid n \in \mathbb{Z}\}$ since $3n+4 = 3(n+1)+1$. Thus, the set of integers which are 1 more than a multiple of 3 is an open ball of radius 1 with center either 1 or 4! In fact, any element of this set is a center!

E8(a) The only uniformly open subsets of $\mathbb{R}^n$ are $\emptyset$ and $\mathbb{R}^n$. To see this, first note that clearly both of these sets are uniformly open. To see the converse, let $U \subseteq \mathbb{R}^n$ be uniformly open; if $U \neq \mathbb{R}^n$, we will show that U is empty. To this end, take $x \in \mathbb{R}^n \backslash U$. Since U is uniformly open, there exists $\varepsilon > 0$ such that for every $u \in U$ we have $B_\varepsilon(u) \subseteq U$.

We claim that $B_\varepsilon(x)$ is disjoint from U. To see this, assume the contrary and take $y \in B_\varepsilon(x) \cap U$. Then since $y \in U$ and U is uniformly open, we have $B_\varepsilon(y) \subseteq U$. But $y \in B_\varepsilon(x) \implies x \in B_\varepsilon(y)$ and thus we deduce x is in U. But $x \in R^n \backslash U$, and this contradiction proves that $B_\varepsilon(x) \cap U = \emptyset$.

Now take any $z \in B_\varepsilon(x)$. Applying the same argument as above (since $z \notin U$) shows that $B_\varepsilon(z)$ is disjoint from U. Thus, $\bigcup\limits_{z \in B_\varepsilon(x)} B_\varepsilon(z)$ is disjoint from U. But this union of balls is simply $B_{2\varepsilon}(x)$, and thus $B_{2\varepsilon}(x)$ is disjoint from U.

Continuing inductively shows that for all $n \geq 1$, we have $B_{n\varepsilon}(x)$ is disjoint from U. But as $n \to \infty$, we have $n\varepsilon \to \infty$, and thus balls of arbitrarily large radius around x are disjoint from U. But this forces U to be empty.

Additional questions:

1. Let (X, d) be a metric space such that X has finitely many points. Prove that for every $x \in X$, the singleton set $\{x\}$ is open.

   Solution: Let $r = \min_{z \in X - \{x\}} d(x, z)$. Note that since X is finite this minimum. Moreover r is positive since each $d(x, z) > 0$ for $z \neq x$. Take any y in $B_r(x)$, and, by definition, $d(y, x) < r$. But then y has distance to x less than the minimum of all points in $X - \{x\}$. This is only possible if $y = x$. Thus, $B_r(x) = \{x\}$ and $\{x\}$ is an open set.

2. Consider the metric space $X = \mathbb{Z}$ endowed with the 3-adic distance function. Determine then open balls $B_1(1)$ and $B_1(4)$.

   Solution: I claim $B_1(1) = \{3n+1 \mid n \in \mathbb{Z}\}$, that is the integers which are 1 more than a multiple of 3. To see this, first note that $d(3n+1, 1) < 1$ for any n. Indeed, if we write $(3n+1) - 1 = 3n$ as $3^k \cdot u$ with u not a multiple of 3, we must have that $k > 0$. Thus $d(3n+1, 1) = (1/3)^k < 1$ and $3n+1 \in B_1(1)$. Hence, $\{3n+1 \mid n \in \mathbb{Z}\} \subseteq B_1(1)$. Conversely, for $x \in B_1(1)$ we will show that $x = 3n+1$ for some n. Indeed, if $d(x, 1) < 1$, then $x - 1 = 3^k u$ with $k > 0$ and thus $x - 1$ is a multiple of 3. In particular, $x - 1 = 3n$ or $x = 3n+1$. Thus, $B_1(1) \subseteq \{3n+1 \mid n \in \mathbb{Z}\}$ and we have show $B_1(1) = \{3n+1 \mid n \in \mathbb{Z}\}$.

   The identical argument shows that $B_1(4) = \{3n+4 \mid n \in \mathbb{Z}\}$.

3. Prove that $(0, \infty)$ is an open set in $\mathbb{R}$ (under the standard metric).

   Solution: Take any $y \in (0, \infty)$. I claim $B_y(y) \subseteq (0, \infty)$ which would prove that $(0, \infty)$ is open. To see this, take $z \in B_y(y)$. Then $|y - z| < y$. But this is only possible if $z > 0$. Thus $B_y(y) \subseteq (0, \infty)$ and $(0, \infty)$ is open.