Topology – MA 564 – Spring 2015 – R. Pollack
HW #2

Complete each of the following exercises.

1. Let (X, d) be a metric space and let $A, B \subseteq X$. Consider each of the following statements. If they are true, prove them. If they are false, give a counter-example.

   (a) $\mathrm{int}(A \cup B) = \mathrm{int}(A) \cup \mathrm{int}(B)$

   (b) $\mathrm{int}(A \cap B) = \mathrm{int}(A) \cap \mathrm{int}(B)$

   (c) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

   (d) $\overline{A \cap B} = \overline{A} \cap \overline{B}$

2. Let (X, d) be a metric space. For $x \in X$ and $r > 0$, we define the *closed ball* $\overline{B}_r(x)$ be to the set $\{z \in X \mid d(z, x) \leq r\}$.

   (a) Prove that $\overline{B}_r(x)$ is a closed set.

   (b) Is it true that $\overline{B_r(x)} = \overline{B}_r(x)$? That is, is the closure of the open ball equal to the closed ball? Either prove this or give a counter-example.

3. Let (X, d) be a metric space. We call a point $x \in X$ a *limit point* of a set $Y \subseteq X$ if for every $r > 0$, we have $B_r(x) \cap Y$ contains a point of Y different from x.

   (a) Clearly, if x is a limit point of Y, then x is adherent to Y. (Make sure you believe this!). Is the converse true? That is, is it true that if x is adherent to Y, then x is a limit point of Y? Again, either prove this or give a counter-example.

   (b) Prove that if x is a limit point of Y, then for every $r > 0$, we have $B_r(x) \cap Y$ is infinite.

4. Let (X, d) be a metric space and let $A, B \subseteq X$. We define the boundary (or frontier) of a set to be

$$\partial A = \overline{A} \cap \overline{X - A}.$$

   (a) In $\mathbb{R}$ under the usual metric, compute $\partial([0, 1))$, $\partial(\mathbb{R})$, $\partial(\mathbb{R} - \{0\})$, $\partial(\mathbb{Q})$, $\partial(\mathbb{Z})$ and $\partial(\emptyset)$. (Here $\mathbb{Q}$ is the set of rational numbers and $\mathbb{Z}$ is the set of integers.)

   Consider each of the following statements. If they are true, prove them. If they are false, give a counter-example.

   (b) $\partial(A \cup B) = \partial A \cup \partial B$

   (c) $\partial(A \cap B) = \partial A \cap \partial B$

   (d) $\partial A = \overline{A} - \mathrm{int}(A)$.

Optional questions (interesting, but not too be turned in):
Freiwald, pg. 82: E11
Gamelin & Greene, chapter 1, section 1: 11,13,14