Topology – MA 564 – Spring 2015 – R. Pollack
HW #2

Complete each of the following exercises.

1. Let (X, d) be a metric space and let $A, B \subseteq X$. Consider each of the following statements. If they are true, prove them. If they are false, give a counter-example.

   (a) $\text{int}(A \cup B) = \text{int}(A) \cup \text{int}(B)$

   Solution: This is false. Consider $X = \mathbb{R}$ with the usual metric. Take $A = [0, 1)$ and $B = [1, 2)$. Then $\text{int}(A) = (0, 1)$ and $\text{int}(B) = (1, 2)$. However, $A \cup B = [0, 2)$ and so $\text{int}(A \cup B) = (0, 2)$. In particular, 1 is in $\text{int}(A \cup B)$ but not in $\text{int}(A) \cup \text{int}(B)$.

   (b) $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$

   Solution: This is true. We first check $\text{int}(A \cap B) \subseteq \text{int}(A) \cap \text{int}(B)$. Take $x \in \text{int}(A \cap B)$. Thus there is some $r > 0$ such that $B_r(x) \subseteq A \cap B$. But then $B_r(x) \subseteq A$ and $B_r(x) \subseteq B$. Hence $x \in \text{int}(A)$ and $x \in \text{int}(B)$ which implies $x \in \text{int}(A) \cap \text{int}(B)$.

   Conversely, take $x \in \text{int}(A) \cap \text{int}(B)$. Since $x \in \text{int}(A)$, there is some $r > 0$ such that $B_r(x) \subseteq A$. Similarly, $x \in \text{int}(B)$ implies there is some $s > 0$ such that $B_s(x) \subseteq B$. Let $t = \min\{r, s\} > 0$. Then $B_t(x) \subseteq B_r(x) \subseteq A$ and $B_t(x) \subseteq B_s(x) \subseteq B$ which implies $B_t(x) \subseteq A \cap B$. Hence, $x \in \text{int}(A \cap B)$ as desired.

   (c) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

   Solution: This is true. First note that if $A_1 \subseteq A_2 \subseteq X$, then $\overline{A}_1 \subseteq \overline{A}_2$. Indeed, if x is adherent to A_1, then x is adherent to A_2 (direct from the definition of adherent). Applying this fact to $A \subseteq A \cup B$, we see that $\overline{A} \subseteq \overline{A \cup B}$. Similarly, $\overline{B} \subseteq \overline{A \cup B}$ and thus $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

   Conversely, $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$ implies $A \cup B \subseteq \overline{A} \cup \overline{B}$. Then by a lemma from class, we have $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$ as $\overline{A} \cup \overline{B}$ is a closed set.

   (d) $\overline{A \cap B} = \overline{A} \cap \overline{B}$

   Solution: This is false. Again take $X = \mathbb{R}$ with the usual metric. Set $A = (0, 1)$ and $B = (1, 2)$. Then $A \cap B = \emptyset$ and hence $\overline{A \cap B} = \emptyset$. However, $\overline{A} = [0, 1]$ and $\overline{B} = [1, 2]$ which implies $\overline{A} \cap \overline{B} = \{1\}$.

2. Let (X, d) be a metric space. For $x \in X$ and $r > 0$, we define the *closed ball* $\overline{B}_r(x)$ be to the set $\{z \in X \mid d(z, x) \leq r\}$.

   (a) Prove that $\overline{B}_r(x)$ is a closed set.

   Solution: To show $\overline{B}_r(x)$ is closed, I'll show the complement is open. To this end, take $z \notin \overline{B}_r(x)$. Set $s = d(z, x) - r$, and I claim $B_s(z) \cap \overline{B}_r(x) = \emptyset$. If not, take w in this intersection. Then $d(z, w) < s = d(z, x) - r$ and $d(w, x) \leq r$. Adding these inequalities gives $d(z, w) + d(w, x) < d(z, x)$ which contradicts the triangle inequality. Thus, $B_s(z) \cap \overline{B}_r(x) = \emptyset$ and $B_s(z)$ is completely contained within the complement of $\overline{B}_r(x)$. Thus this complement is open and $\overline{B}_r(x)$ is closed.

   (b) Is it true that $\overline{B_r(x)} = \overline{B}_r(x)$? That is, is the closure of the open ball equal to the closed ball? Either prove this or give a counter-example.

   Solution: This is false. For instance, take X any set with at least two points and give it the discrete metric. Then for $x \in X$, we have $B_1(x) = \{x\}$ which is a closet set. Thus, the closure of $B_1(x)$ is simply $\{x\}$. However, $\overline{B}_1(x) = \{z \in X \mid d(z, x) \leq 1\} = X$ since distances in the discrete metric are always less than or equal to 1. Since X has at least two points, $\overline{B_1(x)} = \{x\} \neq X = \overline{B}_1(x)$.

Another counter-example: in the 2-adic metric on $\mathbb{Z}$, we have that $B_1(0)$ is the set of even numbers while $B_1(1)$ is the set of odd numbers. Thus the complement of $B_1(0)$ equals $B_1(1)$. Since $B_1(1)$ is open, we then have $B_1(0)$ is closed. Therefore the closure of $B_1(0)$ equals itself and is not equal $\overline{B}_1(0)$ which is the set of all integers.

3. Let (X, d) be a metric space. We call a point $x \in X$ a *limit point* of a set $Y \subseteq X$ if for every $r > 0$, we have $B_r(x) \cap Y$ contains a point of Y different from x.

   (a) Clearly, if x is a limit point of Y, then x is adherent to Y. (Make sure you believe this!). Is the converse true? That is, is it true that if x is adherent to Y, then x is a limit point of Y? Again, either prove this or give a counter-example.

   Solution: This is not true. Consider $X = \mathbb{R}$ with the usual metric and $Y = \{0\}$. Clearly $\{0\}$ is adherent to Y, but it is not a limit point of Y as $B_r(x) \cap \{0\}$ never contains a point different from 0.

   (b) Prove that if x is a limit point of Y, then for every $r > 0$, we have $B_r(x) \cap Y$ is infinite.

   Solution: First solution — assume $B_r(x) \cap Y$ is finite. Pick $s > 0$ small enough so that $d(z, x) > s$ for each of the finitely many $z \in B_r(x) \cap Y$ with $z \neq x$. But then $B_s(x) \cap Y = \{x\}$ contradicting x being a limit point.

   Solution: Second solution — consider $B_r(x) \cap Y$. Since x is a limit point, this intersection contains some point different from x; call this point x_1. Let $r_1 = d(x, x_1)$ which is less than r. Now consider $B_{r_1}(x) \cap Y$ which again contains a point different from x; call this point x_2. Note that $x_2 \neq x_1$ since x_1 is not in $B_{r_1}(x)$ as $d(x, x_1) = r_1$. Continuing this process, we can construct a sequence of distinct points x_n all in $B_r(x) \cap Y$ and thus this set is infinite.

4. Let (X, d) be a metric space and let $A, B \subseteq X$. We define the boundary (or frontier) of a set to be

$$\partial A = \overline{A} \cap \overline{X - A}.$$

   (a) In $\mathbb{R}$ under the usual metric, compute $\partial([0,1))$, $\partial(\mathbb{R})$, $\partial(\mathbb{R} - \{0\})$, $\partial(\mathbb{Q})$, $\partial(\mathbb{Z})$ and $\partial(\emptyset)$. (Here $\mathbb{Q}$ is the set of rational numbers and $\mathbb{Z}$ is the set of integers.)

   Solution:

   i. $\partial([0,1)) = \{0\} \cup \{1\}$
   ii. $\partial(\mathbb{R}) = \emptyset$
   iii. $\partial(\mathbb{R} - \{0\}) = \{0\}$
   iv. $\partial(\mathbb{Q}) = \mathbb{R}$
   v. $\partial(\mathbb{Z}) = \mathbb{Z}$
   vi. $\partial(\emptyset) = \emptyset$.

   Let me explain $\partial(\mathbb{Q}) = \mathbb{R}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we have $\overline{\mathbb{Q}} = \mathbb{R}$. But it is also true that $\mathbb{R} - \mathbb{Q}$ is dense in $\mathbb{R}$, and so $\overline{\mathbb{R} - \mathbb{Q}} = \mathbb{R}$. Note that the statement that $\mathbb{Q}$ is dense in $\mathbb{R}$ is not obvious. It requires the fact that between every two real numbers there is a rational number. Likewise for $\mathbb{R} - \mathbb{Q}$ being dense in $\mathbb{R}$.

   Consider each of the following statements. If they are true, prove them. If they are false, give a counter-example.

   (b) $\partial(A \cup B) = \partial A \cup \partial B$

   Solution: False. $A = [0, 1)$ and $B = [1, 2]$. Then $\partial A = \{0\} \cup \{1\}$ and $\partial B = \{1\} \cup \{2\}$. But $\partial(A \cup B) = \partial([0, 2]) = \{0\} \cup \{2\}$ doesn't contain 1.

(c) $\partial(A \cap B) = \partial A \cap \partial B$

Solution: False. $A = [0, 2)$ and $B = [1, 3]$. Then $\partial A = \{0\} \cup \{1\}$ and $\partial B = \{1\} \cup \{2\}$. But $\partial(A \cup B) = \partial([0, 2]) = \{0\} \cup \{2\}$ doesn't contain 1.

(d) $\partial A = \overline{A} - \text{int}(A)$.

Solution:

$$\begin{aligned}
x \in \overline{A} - \text{int}(A) &\iff x \in \overline{A} \text{ and } x \notin \text{int}(A) \\
&\iff x \in \overline{A} \text{ and for all } r > 0, \text{we have } B_r(x) \not\subseteq A \\
&\iff x \in \overline{A} \text{ and for all } r > 0, \text{we have } B_r(x) \cap (X - A) \neq \emptyset \\
&\iff x \in \overline{A} \text{ and } x \in \overline{X - A} \\
&\iff x \in \overline{A} \cap \overline{X - A} = \partial A
\end{aligned}$$