Topology – MA 564 – Spring 2015 – R. Pollack
HW #3

Complete each of the following exercises.

1. Prove that if U is open and F is closed then $U - F$ is open. Is $F - U$ closed?
2. Prove that in $\mathbb{R}^2$ under the usual metric that the sequence $\{(1/n, 1/n)\}$ converges to $(0, 0)$. Now prove that the same sequence converges to $(0, 0)$ under the taxi-cab metric.
3. Prove that if x is a limit point (defined in HW #2) of a set Y, then there is a sequence $\{x_n\}$ converging to x with $x_n \in Y$ and such that $\{x_n\}$ is not eventually constant.
4. Let $f : X \to Y$ be a function. Let A and B be subsets of X, and let C and D be subsets of Y. Consider each of the following statements. If they are true, prove them. If they are false, give a counter-example.
   (a) $f(A \cup B) = f(A) \cup f(B)$
   (b) $f(A \cap B) = f(A) \cap f(B)$
   (c) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$
   (d) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$
   (e) $f(A - B) = f(A) - f(B)$.
   (f) $f^{-1}(C - D) = f^{-1}(C) - f^{-1}(D)$.
5. Find all isometries from $\mathbb{R}^2 \to \mathbb{R}^2$.
6. Freiwald, Chapter 2: E22, E26, E31
7. Let $f : X \to Y$ be a function. Let $A \subseteq X$ and $B \subseteq Y$. Consider each of the following statements. If they are true, prove them. If they are false, give a counter-example.
   (a) $f(f^{-1}(B)) \subseteq B$
   (b) $f(f^{-1}(B)) \supseteq B$
   (c) $f^{-1}(f(A)) \subseteq A$
   (d) $f^{-1}(f(A)) \supseteq A$
8. Let (X, d) and (Y, s) be metric spaces, and let $f : X \to Y$ be a *continuous* function. Let $A \subseteq X$ and $B \subseteq Y$. Consider each of the following statements. If they are true, prove them. If they are false, give a counter-example.
   (a) $f(\mathrm{int}(A)) \subseteq \mathrm{int}(f(A))$
   (b) $f(\mathrm{int}(A)) \supseteq \mathrm{int}(f(A))$
   (c) $f^{-1}(\mathrm{int}(B)) \subseteq \mathrm{int}(f^{-1}(B))$
   (d) $f^{-1}(\mathrm{int}(B)) \supseteq \mathrm{int}(f^{-1}(B))$
   (e) $f(\overline{A}) \subseteq \overline{f(A)}$
   (f) $f(\overline{A}) \supseteq \overline{f(A)}$
   (g) $f^{-1}(\overline{B}) \subseteq \overline{f^{-1}(B)}$
   (h) $f^{-1}(\overline{B}) \supseteq \overline{f^{-1}(B)}$
9. Let (X, d) be a metric space and fix $\alpha \in X$. Consider the function $f : X \to \mathbb{R}$ defined by $f(x) = d(x, \alpha)$, in words the "distance to α" function. Prove that f is a continuous function.