Topology – MA 564 – Spring 2015 – R. Pollack
HW #3 Solutions

Complete each of the following exercises.

1. Prove that if U is open and F is closed then $U - F$ is open. Is $F - U$ closed?

   Solution: We first show $U - F$ is open. Take $x \in U - F$. Then since U is open there exists $r > 0$ such that $B_r(x) \subseteq U$. Since F is closed and $x \notin F$, there is some $s > 0$ such that $B_s(x) \cap F = \emptyset$. If $t = \min\{r, s\} > 0$, then $B_t(x) \subseteq U$ and $B_t(x) \cap F = \emptyset$; thus, $B_t(x) \subseteq U - F$ and we deduce that $U - F$ is open.

   Here's a second proof. By definition, $U - F = U \cap F^c$ where F^c is the complement of F. Since F is closed, F^c is open. Thus $U \cap F^c$ is open (as intersections of opens are open), and we deduce $U - F$ is open.

   It is also true that $F - U$ is closed. Again $F - U = F \cap U^c$ which is clearly closed as F is closed and U^c is closed.

2. Prove that in $\mathbb{R}^2$ under the usual metric that the sequence $\{(1/n, 1/n)\}$ converges to $(0, 0)$. Now prove that the same sequence converges to $(0, 0)$ under the taxi-cab metric.

   Solution: Take $\varepsilon > 0$. Then set $N = \frac{\sqrt{2}}{\varepsilon}$. For $n > N$, we have

$$n > \frac{\sqrt{2}}{\varepsilon} \implies n^2 > \frac{2}{\varepsilon^2} \implies \frac{2}{n^2} < \varepsilon^2 \implies \frac{1}{n^2} + \frac{1}{n^2} < \varepsilon^2$$
$$\implies \sqrt{\frac{1}{n^2} + \frac{1}{n^2}} < \varepsilon \implies d\left((1/n, 1/n), (0, 0)\right) < \varepsilon.$$

   Thus, $\{(1/n, 1/n)\}$ converges to $(0, 0)$ under the usual Euclidean metric on $\mathbb{R}^2$.

   For the taxi-cab metric, fix $\varepsilon > 0$ and set $N = \frac{2}{\varepsilon}$. Then, for $n > N$, we have

$$n > \frac{2}{\varepsilon} \implies \frac{2}{n} < \varepsilon \implies \frac{1}{n} + \frac{1}{n} < \varepsilon \implies \left(\frac{1}{n} - 0\right) + \left(\frac{1}{n} - 0\right) < \varepsilon \implies d_{\text{taxi}}\left((1/n, 1/n), (0, 0)\right) < \varepsilon.$$

   Thus, $\{(1/n, 1/n)\}$ converges to $(0, 0)$ under the taxi-cab metric on $\mathbb{R}^2$.

3. Prove that if x is a limit point (defined in HW #2) of a set Y, then there is a sequence $\{x_n\}$ converging to x with $x_n \in Y$ and such that $\{x_n\}$ is not eventually constant.

   Solution: For each $n \geq 1$, consider $B_{1/n}(x)$. Since x is a limit point, we have $B_{1/n}(x) \cap Y$ contains some point different from x. Pick such a point and call it x_n. Then (as proved in class) $\{x_n\}$ converges to x. We just need to check that $\{x_n\}$ is not eventually constant. To see this, assume the opposite — that is, assume that there is some $y \in X$ and some N such that if $n > N$ then $x_n = y$. Then $y = x_n \in B_{1/n}(x)$ for every $n > N$, and hence $d(x, y) < 1/n$ for all $n > N$. But then $d(x, y) = 0$ and $y = x$. This is a contraction since by construction $y = x_n \neq x$.

4. Let $f : X \to Y$ be a function. Let A and B be subsets of X, and let C and D be subsets of Y. Consider each of the following statements. If they are true, prove them. If they are false, give a counter-example.

   (a) $f(A \cup B) = f(A) \cup f(B)$

   Solution: True.

$$\begin{aligned} y \in f(A \cup B) &\iff y = f(x) \text{ with } x \in A \cup B \\ &\iff y = f(x) \text{ with } x \in A \text{ or } x \in B \\ &\iff y \in f(A) \text{ or } y \in f(B) \\ &\iff y \in f(A) \cup f(B) \end{aligned}$$

(b) $f(A \cap B) = f(A) \cap f(B)$

Solution: False! Take $X = Y = \mathbb{R}$ and $f(x) = x^2$. Let $A = (-\infty, 0]$ and $B = [0, \infty)$. Then $f(A) = [0, \infty) = f(B)$ and so $f(A) \cap f(B) = [0, \infty)$. However, $A \cap B = \{0\}$ and so $f(A \cap B) = \{0\}$.

(c) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$

Solution: True.

$$\begin{aligned} x \in f^{-1}(C \cup D) &\iff f(x) \in C \cup D \\ &\iff f(x) \in C \text{ or } f(x) \in D \\ &\iff x \in f^{-1}(C) \text{ or } x \in f^{-1}(D) \\ &\iff x \in f^{-1}(C) \cup f^{-1}(D) \end{aligned}$$

(d) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

Solution: True.

$$\begin{aligned} x \in f^{-1}(C \cap D) &\iff f(x) \in C \cap D \\ &\iff f(x) \in C \text{ and } f(x) \in D \\ &\iff x \in f^{-1}(C) \text{ and } x \in f^{-1}(D) \\ &\iff x \in f^{-1}(C) \cap f^{-1}(D) \end{aligned}$$

(e) $f(A - B) = f(A) - f(B)$.

Solution: False. Take $X = Y = \mathbb{R}$ and $f(x) = x^2$. Let $A = \mathbb{R}$ and $B = [0, \infty)$. Then $A - B = (-\infty, 0)$ and so $f(A - B) = (0, \infty)$. However, $f(A) = f(\mathbb{R}) = [0, \infty)$ and $f(B) = [0, \infty)$ so that $f(A) - f(B) = \emptyset$.

(f) $f^{-1}(C - D) = f^{-1}(C) - f^{-1}(D)$.

Solution: True.

$$\begin{aligned} x \in f^{-1}(C - D) &\iff f(x) \in C - D \\ &\iff f(x) \in C \text{ and } f(x) \notin D \\ &\iff x \in f^{-1}(C) \text{ and } x \notin f^{-1}(D) \\ &\iff x \in f^{-1}(C) - f^{-1}(D) \end{aligned}$$

6. Freiwald, Chapter 2: E22, E26, E31

E26 *Solution*: Note that $\chi_A^{-1}(\{1\}) = A$ and moreover, $A = \chi_A^{-1}(B_{1/2}(1))$. If χ_A is continuous, then this pre-image is open as $B_{1/2}(1)$ is open, and thus A is open. The same argument using $B_{1/2}(0)$ shows that A^c is open and thus A is closed. Hence, if χ_A is continuous, then A is clopen (closed and open). The only subsets of $\mathbb{R}$ which are clopen are $\emptyset$ and $\mathbb{R}$.

E31(a) *Solution*: Assume $\mathbb{R}$ and $\mathbb{R}^2$ are isometric with $f : \mathbb{R}^2 \to \mathbb{R}$ being an isometry. Then all points on the unit circle of $\mathbb{R}^2$ need to map to points at distance 1 from $f((0,0))$. But in $\mathbb{R}$ there are only two points at distance 1 from any given point. Since the unit circle is infinite this means that f cannot be one-to-one which is a contradiction.

E31(b) *Solution*: Clearly, $f(x) = x$ is an isometry that fixes a. Further, $f(x) = 2a - x$ is also an isometry (think of it as first negating x and then translating by $2a$) which fixes a. To see that these are the only isometries, first consider $f(a+1)$. We must have that $f(a+1)$ is at distance 1 from a and is thus $a+1$ or $a-1$.

Let's first consider the case where $f(a+1) = a+1$. Now consider $f(a+b)$ where $b \neq 0, 1$. Then, since $a+b$ has distance b from a we must have that $f(a+b)$ has distance b from $f(a) = a$. In particular, $f(a+b) = a+b$ or $a-b$. But $a+b$ also has distance $b-1$ from $a+1$. Thus $f(a+b)$ has distance $b-1$ from $f(a+1) = a+1$ which implies $f(a+b) = a+1+b-1 = a+b$ or $a+1-(b-1) = 2+a-b$. Thus, the only possibility is $f(a+b) = a+b$ and thus $f(x) = x$ for all x.

Let's now consider the case where $f(a+1) = a-1$. Now consider $f(a+b)$ where $b \neq 0, 1$. Then, since $a+b$ has distance b from a we must have that $f(a+b)$ has distance b from $f(a) = a$. In particular, $f(a+b) = a+b$ or $a-b$. But $a+b$ also has distance $b-1$ from $a+1$. Thus $f(a+b)$ has distance $b-1$ from $f(a+1) = a-1$ which implies $f(a+b) = a-1+b-1 = a+b-2$ or $a-1-(b-1) = a-b$. Thus, the only possibility is $f(a+b) = a-b$ and thus $f(x) = 2a-x$ for all x.

E31(c) *Solution*: Let $X = [0,\infty)$ with the usual metric. Then $f : X \to X$ by $f(x) = x+1$ is an isometry of X with a proper subset of itself.

7. Let $f : X \to Y$ be a function. Let $A \subseteq X$ and $B \subseteq Y$. Consider each of the following statements. If they are true, prove them. If they are false, give a counter-example.

   (a) $f(f^{-1}(B)) \subseteq B$

   Solution: True. For $y \in f(f^{-1}(B))$, we have $y = f(x)$ for some $x \in f^{-1}(B)$. Thus, $y = f(x) \in B$.

   (b) $f(f^{-1}(B)) \supseteq B$

   Solution: False. Take $X = Y = \mathbb{R}$ and $f(x) = x^2$. Let $B = \mathbb{R}$. Then $f^{-1}(B) = \mathbb{R}$, and $f(f^{-1}(B)) = f(\mathbb{R}) = [0,\infty) \not\supseteq B$.

   (c) $f^{-1}(f(A)) \subseteq A$

   Solution: False. Take $X = Y = \mathbb{R}$ and $f(x) = x^2$. Let $A = [0,\infty)$. Then $f(A) = [0,\infty)$, and $f^{-1}(f(A)) = \mathbb{R} \not\subseteq A$.

   (d) $f^{-1}(f(A)) \supseteq A$

   Solution: True. For $x \in A$, to check if $x \in f^{-1}(f(A))$ we simply need to see if $f(x) \in f(A)$ which is obviously true.

8. Let (X,d) and (Y,s) be metric spaces, and let $f : X \to Y$ be a *continuous* function. Let $A \subseteq X$ and $B \subseteq Y$. Consider each of the following statements. If they are true, prove them. If they are false, give a counter-example.

   (a) $f(\text{int}(A)) \subseteq \text{int}(f(A))$

   Solution: False. Take $X = Y = \mathbb{R}$ and let $f(x)$ be the constant function which is identically 0. Let $A = \mathbb{R}$. Then $\text{int}(A) = \mathbb{R}$ and $f(\text{int}(A)) = \{0\}$. However, $f(A) = \{0\}$ and $\text{int}(f(A)) = \emptyset$.

(b) $f(\text{int}(A)) \supseteq \text{int}(f(A))$

Solution: False. Take $X = \mathbb{R}^2$, $Y = \mathbb{R}$ and let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x$ (which is a continuous function). Let $A = \{(x, 0) \in \mathbb{R}^2\}$. Then $\text{int}(A) = \emptyset$ and so $f(\text{int}(A)) = \emptyset$. However, $f(A) = \mathbb{R}$ so that $\text{int}(f(A)) = \mathbb{R}$.

(c) $f^{-1}(\text{int}(B)) \subseteq \text{int}(f^{-1}(B))$

Solution: True. Take $x \in f^{-1}(\text{int}(B))$. Then $f(x) \in \text{int}(B)$. So there exists $\varepsilon > 0$ such that $B_\varepsilon(f(x)) \subseteq B$. By continuity, there is some $\delta > 0$ such that $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x))) \subseteq f^{-1}(B)$. Thus, $x \in \text{int}(f^{-1}(B))$.

(d) $f^{-1}(\text{int}(B)) \supseteq \text{int}(f^{-1}(B))$

Solution: False. Take $X = Y = \mathbb{R}$ and let f be the constant function sending everything to 0. Let $B = \{0\}$. Then $\text{int}(B) = \emptyset$ and so $f^{-1}(\text{int}(B)) = \emptyset$. However, $f^{-1}(B) = \mathbb{R}$ and thus $\text{int}(f^{-1}(B)) = \mathbb{R}$.

(e) $f(\overline{A}) \subseteq \overline{f(A)}$

Solution: True. Take $y \in f(\overline{A})$. Then there is some $x \in \overline{A}$ with $f(x) = y$. As proven in class, there is some sequence $\{x_n\}$ converging to x with each $x_n \in A$. Then, by the continuity of f, we have $\{f(x_n)\}$ converges to $f(x)$. Hence, $y = f(x) \in \overline{f(A)}$ since each $f(x_n) \in f(A)$.

(f) $f(\overline{A}) \supseteq \overline{f(A)}$

Solution: False. Take $X = \mathbb{R}^2$ and $Y = \mathbb{R}$ and $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = x$. Let A by the hyperbola, $\{(x, y) \in \mathbb{R}^2 \mid xy = 1\}$. Then $f(A) = \mathbb{R} - \{0\}$ and $\overline{f(A)} = \mathbb{R}$. But A is closed and thus $\overline{A} = A$. We then have $f(\overline{A}) = f(A) = \mathbb{R} - \{0\} \not\supseteq \overline{f(A)} = \mathbb{R}$.

(g) $f^{-1}(\overline{B}) \subseteq \overline{f^{-1}(B)}$

Solution: False. Take $X = \mathbb{R}$ and $Y = \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ by $f(x, y) = 0$. Let $B = (0, 1)$. Then $\overline{B} = [0, 1]$ and $f^{-1}(\overline{B}) = \mathbb{R}$. However, $f^{-1}(B) = \emptyset$ and thus $\overline{f^{-1}(B)} = \emptyset$.

(h) $f^{-1}(\overline{B}) \supseteq \overline{f^{-1}(B)}$

Solution: True. Take $x \in \overline{f^{-1}(B)}$. Then (by a theorem in class) there exists a sequence $\{x_n\} \to x$ with each $x_n \in f^{-1}(B)$ (i.e. $f(x_n) \in B$). Since f is continuous, we have $\{f(x_n)\} \to f(x)$ which implies $f(x) \in \overline{B}$ and thus $x \in f^{-1}(\overline{B})$.

9. Let (X, d) be a metric space and fix $\alpha \in X$. Consider the function $f : X \to \mathbb{R}$ defined by $f(x) = d(x, \alpha)$, in words the "distance to α" function. Prove that f is a continuous function.

Solution: Take $a \in X$ and fix $\varepsilon > 0$. Set $\delta = \varepsilon$. Then for x satisfying $d(x, a) < \delta = \varepsilon$, we have

$$d(x, \alpha) \leq d(x, a) + d(a, \alpha)$$

and thus

$$d(x, \alpha) - d(a, \alpha) \leq d(x, a).$$

Likewise,

$$d(a, \alpha) \leq d(x, a) + d(x, \alpha)$$

and thus

$$d(a, \alpha) - d(x, \alpha) \leq d(x, a).$$

Putting these together gives

$$|d(a, \alpha) - d(x, \alpha)| \leq d(x, a) < \delta = \varepsilon$$

as desired.