Topology – MA 564 – Spring 2015 – R. Pollack
HW #4

Complete each of the following exercises.

1. Let $X = \mathbb{R}$ and set $\mathcal{T} := \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$.

   (a) Prove that $\mathcal{T}$ forms a topology on $\mathbb{R}$. We call this the "right ray topology".

   (b) Is $(\mathbb{R}, \mathcal{T})$ a T_0 space? a T_1 space? a T_2 space?

   (c) Is $(\mathbb{R}, \mathcal{T})$ metrizable?

   (d) For $\alpha \in \mathbb{R}$, what is $\overline{\{\alpha\}}$? That is, what is the closure of a singleton set?

   (e) Consider the sequence $\{1, 2, 3, 4, \dots\}$. What elements of $\mathbb{R}$ does this sequence converge to in the right ray topology?

   (f) Consider the sequence $\{-1, -2, -3, -4, \dots\}$. What elements of $\mathbb{R}$ does this sequence converge to in the right ray topology?

   (g) Consider the sequence $\{0, 0, 0, 0, \dots\}$. What elements of $\mathbb{R}$ does this sequence converge to in the right ray topology?

2. Let X be any set and put $\mathcal{T} := \{X - \{\alpha_1, \dots, \alpha_n\}\} \cup \{\varnothing\}$, that is, $\mathcal{T}$ is the collection of sets with finite complement together with the empty set.

   (a) Prove that $\mathcal{T}$ forms a topology on X. We call this the "cofinite topology" on X.

   (b) Is $(X, \mathcal{T})$ a T_0 space? a T_1 space? a T_2 space?

   (c) Is $(X, \mathcal{T})$ metrizable?

   (d) For $\alpha \in X$, what is $\overline{\{\alpha\}}$? That is, what is the closure of a singleton set?

   (e) Consider an arbitrary sequence $\{x_n\}$. What elements of X does this sequence converge to in cofinite topology? Does it depend on whether X is infinite or not?

3. Let $(X, \mathcal{T})$ be a topological space and let $Y \subseteq X$. We endow Y with the subspace topology to make it into a topology space.

   (a) Let $S \subseteq X$ and write $\overline{S \cap Y}^{Y}$ for the closure of $S \cap Y$ as a subset of Y under the subspace topology. Prove that $\overline{S \cap Y}^{Y} \subseteq \overline{S} \cap Y$.

   (b) Find an example where $\overline{S \cap Y}^{Y}$ is strictly smaller than $\overline{S} \cap Y$.

   (c) If $U \subseteq Y$ is open in Y (under the subspace topology) is U necessarily open in X?

4. Freiwald, Chapter 3: E4(a,b,c,d), E6, E8

5. Let X and Y be two topological spaces and let $f : X \to Y$ be a continuous map.

   (a) Prove that if $F \subseteq Y$ is closed, then $f^{-1}(F)$ is closed in X.

   (b) If Z is another topological space and $g : Y \to Z$ is continuous, prove that the composite $g \circ f : X \to Z$ is continuous.