Topology – MA 564 – Spring 2015 – R. Pollack
HW #4

Complete each of the following exercises.

1. Let $X = \mathbb{R}$ and set $\mathcal{T} := \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$.

   (a) Prove that $\mathcal{T}$ forms a topology on $\mathbb{R}$. We call this the "right ray topology".

   Solution: By definition $\varnothing$ and $\mathbb{R}$ are open. To check unions of opens are open, let $\{U_\alpha\}_{\alpha \in I}$ denote some collection of open sets where $U_\alpha = (\alpha, \infty)$. Then if $s = \inf\limits_{\alpha \in I} \alpha$ (where we allow $s = -\infty$ as a possibility), then

   $$\bigcup_{\alpha \in I} (\alpha, \infty) = (s, \infty)$$

   which is again an open set.
   For intersections, we have

   $$\bigcap_{k=1}^{n} (\alpha_k, \infty) = (s, \infty)$$

   where $s = \max\limits_k \alpha_k$.

   (b) Is $(\mathbb{R}, \mathcal{T})$ a T_0 space? a T_1 space? a T_2 space?

   Solution: This is a T_0-space since given $x < y$, we have $(\frac{x+y}{2}, \infty)$ is an open containing y but not x. However, it is not a T_1-space (and thus not a T_2-space). Indeed, if $x \in (\alpha, \infty)$, then since $y > x$, we have y is also in (α, ∞). Thus any open which contains x automatically contains y.

   (c) Is $(\mathbb{R}, \mathcal{T})$ metrizable?

   Solution: No. Any metrizable space is T_2.

   (d) For $\alpha \in \mathbb{R}$, what is $\overline{\{\alpha\}}$? That is, what is the closure of a singleton set?

   Solution: We have

   $$\overline{\{\alpha\}} = \bigcap_F F$$

   where F runs through all closed sets containing α. Closed sets in this topology are left closed rays (e.g. $(-\infty, \alpha]$), and thus closed sets containing α are of the form $(-\infty, \beta]$ with $\beta \geq \alpha$. Thus

   $$\overline{\{\alpha\}} = \bigcap_{\beta \geq \alpha} (-\infty, \beta] = (-\infty, \alpha].$$

   (e) Consider the sequence $\{1, 2, 3, 4, \ldots\}$. What elements of $\mathbb{R}$ does this sequence converge to in the right ray topology?

   Solution: This sequence converges to every $x \in \mathbb{R}$. To see this, take some open U containing x. Then $U = (\alpha, \infty)$ with $\alpha < x$. Thus, once n is large enough $n \in U$ which proves that $\{n\} \to x$.

   (f) Consider the sequence $\{-1, -2, -3, -4, \ldots\}$. What elements of $\mathbb{R}$ does this sequence converge to in the right ray topology?

   Solution: This sequence is not convergent. To see this, assume $\{-n\} \to x$. Take $U = (x - 1, \infty)$. But it is not true that our sequence will be in this open if we go far enough out. Indeed, if we go far enough out, we can guarantee that the sequence is not in U.

(g) Consider the sequence $\{0, 0, 0, 0, \dots\}$. What elements of $\mathbb{R}$ does this sequence converge to in the right ray topology?

Solution: This sequence converges to all $x \leq 0$. To see this, take $U = (y, \infty)$ containing x. Thus $y < x$ and so $y < 0$. In particular, $0 \in U$ which implies the constant sequence 0 converges to x.

2. Let X be any set and put $\mathcal{T} := \{X - \{\alpha_1, \dots, \alpha_n\}\} \cup \{\varnothing\}$, that is, $\mathcal{T}$ is the collection of sets with finite complement together with the empty set.

(a) Prove that $\mathcal{T}$ forms a topology on X. We call this the "cofinite topology" on X.

Solution: Clearly, $\varnothing$ and X are open. To check unions, if U_α are all open, then

$$\left(\bigcup_\alpha U_\alpha\right)^c = \bigcap_\alpha U_\alpha^c.$$

Since each U_α is cofinite, we have that each U_α^c is finite and thus their intersection is finite. (In fact, we only needed that a single U_α was cofinite.) Thus, $\bigcup_\alpha U_\alpha$ is cofinite.
To check intersections, if $U_1, \dots, U_n$ are all open, then

$$\left(\bigcap_{i=1}^n U_i\right)^c = \bigcup_{i=1}^n U_i^c.$$

Since each U_i is cofinite, each U_i^c is finite and thus their union is finite (since the union is a finite union). Thus $\bigcap_{i=1}^n U_i$ is cofinite.

(b) Is $(X, \mathcal{T})$ a T_0 space? a T_1 space? a T_2 space?

Solution: This space is T_1 (and thus T_0). Indeed, given $x \neq y$, then $X - \{y\}$ is an open containing x but not y.
If the space is finite, then it is discrete and hence T_2. If the space is infinite, then there aren't any disjoint opens and thus the space is not T_2.

(c) Is $(X, \mathcal{T})$ metrizable?

Solution: No. Metrizable spaces are T_2.

(d) For $\alpha \in X$, what is $\overline{\{\alpha\}}$? That is, what is the closure of a singleton set?

Solution: The point $\{\alpha\}$ is closed since it's complement is cofinite. Thus, $\overline{\{\alpha\}} = \alpha$.

(e) Consider an arbitrary sequence $\{x_n\}$. What elements of X does this sequence converge to in cofinite topology? Does it depend on whether X is infinite or not?

Solution: If X is finite, then the cofinite topology is simply the discrete topology. In particular, the sequence $\{x_n\}$ converges iff it is eventually constant.
If X is infinite, then $\{x_n\}$ converges to every point in the space. Indeed, take an arbitrary $x \in X$ and take an open U containing x. Then U is cofinite and so it includes all but finitely many elements of our given sequence. Thus, for n large enough, $x_n \in U$ which proves $\{x_n\} \to x$.

3. Let $(X, \mathcal{T})$ be a topological space and let $Y \subseteq X$. We endow Y with the subspace topology to make it into a topology space.

(a) Let $S \subseteq X$ and write $\overline{S \cap Y}^Y$ for the closure of $S \cap Y$ as a subset of Y under the subspace topology. Prove that $\overline{S \cap Y}^Y \subseteq \overline{S} \cap Y$.

Solution: We have $\overline{S \cap Y}^Y$ equals the intersection of all closed subsets of Y containing $S \cap Y$. Thus, it suffices to see that $\overline{S} \cap Y$ is a closed subset of Y. To see this, note that

$$Y - (\overline{S} \cap Y) = (X - \overline{S}) \cap Y.$$

Since $X - \overline{S}$ is open in X, we have $(X - \overline{S}) \cap Y$ is open in the subspace topology of Y. Thus $Y - (\overline{S} \cap Y)$ is open and hence $\overline{S} \cap Y$ is closed as desired.

(b) Find an example where $\overline{S \cap Y}^Y$ is strictly smaller than $\overline{S} \cap Y$.

Solution: Take $X = \mathbb{R}$, $Y = \mathbb{R} - \mathbb{Q}$, and $S = \mathbb{Q}$. Then $\overline{S} = \mathbb{R}$ so $\overline{S} \cap Y = \mathbb{R} - \mathbb{Q}$. However, $S \cap Y = \varnothing$ whose closure is also empty.

(c) If $U \subseteq Y$ is open in Y (under the subspace topology) is U necessarily open in X?

Solution: Definitely not. Take $X = \mathbb{R}^2$ and Y equal to the x-axis. Then any non-empty open subset of Y is not open in X.

4. Freiwald, Chapter 3: E4(a,b,c,d), E6, E8

Solution: E4(a): The right ray topology is not T_1 as described above.

Solution: E4(b): In this question, we are defining T_1 as "points are closed". I'll check here that this definition is equivalent to the one we gave in class.

Let's first check that the definition I gave in class implies points are closed. To this end, we check that $\{x\}^c$ is open. For any $y \neq x$, by definition in class, there is some open U_y which contains y but not x. But then $\cup_{y \neq x} U_y = X - \{x\}$ and hence $\{x\}^c$ is open.

Conversely, we check that "points are closed" implies the definition of T_1 I gave in class. To this end, take $x \neq y$. Then since $\{x\}$ is closed, we have $\{x\}^c$ is open and this is an open which contains y but not x.

Solution: E4(c): For every $y \neq x$, let U_y be an open containing x but not y. Then $\cap_{y \neq x} U_y$ is an intersection of open sets. Clearly x is in this intersection as it is in every U_y. However, no other point can be in the intersection as $y \notin U_y$. Thus $\cap_{y \neq x} U_y = \{x\}$.

Solution: E4(d): Let X be T_1 and $Y \subseteq X$. Take $x \neq y$ in Y. Then since X is T_1 there exists U open in X such that $x \in U$ and $y \notin U$. We then have that $U \cap Y$ is open in Y and $x \in U \cap Y$ while $y \notin U \cap Y$. Thus, Y is T_1.

5. Let X and Y be two topological spaces and let $f : X \to Y$ be a continuous map.

(a) Prove that if $F \subseteq Y$ is closed, then $f^{-1}(F)$ is closed in X.

Solution: We have

$$X - f^{-1}(F) = f^{-1}(Y - F).$$

Since F is closed, $Y - F$ is open and thus by definition $f^{-1}(Y - F)$ is open since f is continuous. Hence $X - f^{-1}(F)$ is open which implies $f^{-1}(F)$ is closed.

(b) If Z is another topological space and $g : Y \to Z$ is continuous, prove that the composite $g \circ f : X \to Z$ is continuous.

Solution: Take U open in Z. We need to check $(g \circ f)^{-1}(U)$ is open in X. But

$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)).$$

Since U is open and g is continuous, we have $g^{-1}(U)$ is open. Since f is continuous, we have $f^{-1}(g^{-1}(U))$ is continuous. Thus, $(g \circ f)^{-1}(U)$ is open and $g \circ f$ is continuous.