Topology – MA 564 – Spring 2015 – R. Pollack
HW #5

Complete each of the following exercises.

1. Let X be a set and equip it with the cofinite topology. Prove that X is compact.
2. Let X be a discrete topology space. Find a necessary and sufficient condition on X so that X is compact.
3. Determine whether each of the following subsets of $\mathbb{R}$ are compact under the right-ray topology. Justify your answer:
   (a) $[0, 1]$
   (b) $(0, 1]$
   (c) $[0, 1)$
4. Let X be a topological space with the following property: if $\{F_\alpha\}$ is a collection of closed subsets such any finite intersection of the F_α is non-empty, then $\cap_\alpha F_\alpha$ is non-empty. Prove that X is compact. (Hint: take complements and think about open covers.)
5. Let K be a compact subset of $\mathbb{R}^n$ (under the standard topology). Prove that K is closed.
   (Hint: If K is not closed, there is some limit point x of K not in K. Show that $\{B_{1/m}(x)^c\}_{m\geq 1}$ is an open cover of K with no finite subcover.)
6. Let K be a compact subset of $\mathbb{R}^n$ (under the standard topology). Prove that K is bounded.
   (Hint: If K were unbounded, then $B_N(0)$ would not contain K for any $N \geq 0$.)
7. Let X be a T_2 topological space and let K_1 and K_2 be compact subsets. Prove that $K_1 \cap K_2$ is compact. Does this remain true for infinite intersections?

Additional questions:

1. Does question 7 still hold if we no longer assume that X is T_2?
2. Let K_1 and K_2 be compact subsets of a T_2 topology space X. Find disjoint open sets U and V such that $K_1 \subseteq U$ and $K_2 \subseteq V$.
   (Hint: use the Theorem proven in class that a point and a compact set can be separated by open sets.)