Topology – MA 564 – Spring 2015 – R. Pollack
HW #5

Complete each of the following exercises.

1. Let X be a set and equip it with the cofinite topology. Prove that X is compact.

   Solution: Let $\{U_\alpha\}$ denote an open cover of X. Pick a single non-empty element of this cover, U_α. Then $X - U_\alpha$ is finite, say equal to $\{x_1, \ldots, x_n\}$. Each of the x_i is contained in some U_{α_i} and thus U_α together with $U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}$ is a finite subcover.

2. Let X be a discrete topology space. Find a necessary and sufficient condition on X so that X is compact.

   Solution: X is compact in this case if and only if it is finite. Clearly if X is finite then it is compact (since every open cover is automatically finite). Conversely, if X is compact and discrete, then the singleton sets $\{x\}$ as x varies over X is an open cover of X. Since X is compact, this cover has a finite subcover. But clearly this cover has no proper subcovers (since if we remove any open we would no longer have a cover). This implies then that our original cover has only finitely many opens and thus X is finite.

3. Determine whether each of the following subsets of $\mathbb{R}$ are compact under the right-ray topology. Justify your answer:

   (a) $[0, 1]$
   (b) $(0, 1]$
   (c) $[0, 1)$

   Solution: (b) is not compact because the open cover $\{(1/n, \infty)\}$ has no finite subcover. Both of (a) and (c) are compact for the same reason. Take an open cover of either space. Take an open in this cover that contains the element 0. But then, by the definition of this topology, this single open contains $[0, \infty)$ and thus the entire space. Hence we not only have a finite subcover, we have a subcover with only 1 open!

4. Let X be a topological space with the following property: if $\{F_\alpha\}$ is a collection of closed subsets such any finite intersection of the F_α is non-empty, then $\cap_\alpha F_\alpha$ is non-empty. Prove that X is compact. (Hint: take complements and think about open covers.)

   Solution: Let $\{U_\alpha\}$ denote an open cover of X. The contrapositive of our assumption is that if we have a collection of closed sets so that their intersection is empty, then there is some finite intersection in this collection that is empty. We will apply this to the collection of closed sets $\{X - U_\alpha\}$. Indeed,

   $$\bigcap_\alpha X - U_\alpha = X - \bigcup_\alpha U_\alpha = X - X = \varnothing.$$

   Thus, we know that there are some $\alpha_1, \ldots, \alpha_n$, such that $\displaystyle\bigcap_{i=1}^n X - U_{\alpha_i} = \varnothing$. But then

   $$\varnothing = \bigcap_{i=1}^n X - U_{\alpha_i} = X - \bigcup_{i=1}^n U_{\alpha_i}$$

   which implies $X = \bigcup_{i=1}^n U_{\alpha_i}$ and $U_{\alpha_1}, \ldots, U_{\alpha_n}$ is a finite subcover.

5. Let K be a compact subset of $\mathbb{R}^n$ (under the standard topology). Prove that K is closed.
(Hint: If K is not closed, there is some limit point x of K not in K. Show that $\{\overline{B}_{1/m}(x)^c\}_{m\geq 1}$ is an open cover of K with no finite subcover.)

   Solution: Assume K is not closed. Then there is some limit point x of K not in K. We claim then that $\{\overline{B}_{1/m}(x)^c\}$ is an open cover with no finite subcover. To see that it is a cover, take any $z \in K$. Then since $x \notin K$, the distance between z and x is positive. Thus, for some m, $||x - z|| > 1/n$ and z is in $\overline{B}_{1/m}(x)^c$ as desired.

   To see that this cover has no finite subcover, assume that one exists. Since this cover is a sequence of increasing opens, if there were a finite subcover, then there would exist some m such that $K \subseteq \overline{B}_{1/m}(x)^c$. But since x is a limit point of K, there is some $z \in K$ such that $||z - x|| < 1/m$ which is a contradiction.

   Thus K is closed.

6. Let K be a compact subset of $\mathbb{R}^n$ (under the standard topology). Prove that K is bounded.
(Hint: If K were unbounded, then $B_N(0)$ would not contain K for any $N \geq 0$.)

   Solution: Consider the open cover of K given by $\{B_N(0)\}$ as N varies over all positive integers. This is clearly a cover as the union of these balls covers all of $\mathbb{R}^n$. Since K is compact, this cover has a finite subcover. But since this cover is a collection of increasing opens, this means that there is a single open $B_N(0)$ which contains K. But this exactly means that K is bounded.

7. Let X be a T_2 topological space and let K_1 and K_2 be compact subsets. Prove that $K_1 \cap K_2$ is compact. Does this remain true for infinite intersections?

   Solution: Since X is T_2 both K_1 and K_2 are closed as they are compact. Thus $K_1 \cap K_2$ is closed. But then the closed $K_1 \cap K_2$ is contained in the compact K_1, and is thus itself compact.

   This does not hold for infinite intersections. For instance, in $\mathbb{R}$, we have $\cap_{n=1}^{\infty}[-1/n, 1 + 1/n] = (0, 1)$ which is not compact.