Topology – MA 564 – Spring 2015 – R. Pollack
HW #6

Complete each of the following exercises.

1. Let X be a topological space. Prove that the following two properties are equivalent.

   - If $U \subseteq X$ is clopen, then $U = \varnothing$ or $U = X$.
   - Whenever $X = U \cup V$ with U, V disjoint open sets, then either $U = \varnothing$ or $V = \varnothing$.

   A topological space that satisfies either of these equivalent conditions is called *connected*.

   Solution: We first check the first condition implies the second. So write $X = U \cup V$ with U, V disjoint open sets. Then $U^c = V$ and so U is both open and closed. Hence, by the first condition $U = \varnothing$ or X. If $U = X$ we are done. If $U = \varnothing$ then $V = X$ and we are done.

   Conversely, we assume the second condition and derive the first. To this end, take $U \subseteq X$ clopen. Since U is clopen, U^c is open. Then $X = U \cup U^c$ with U and U^c disjoint opens. Hence, $U = \varnothing$ or $U^c = \varnothing$ which implies $U = \varnothing$ or X.

2. Let X be an infinite set with the cofinite topology. Is X connected?

   Solution: Yes, X is connected. Indeed, if $U \subseteq X$ is a non-empty clopen, then U^c is finite and open. But finite sets in the cofinite topology (for an infinite space) are open iff they are empty. Thus $U^c = \varnothing$ and $U = X$ as desired.

3. We define an *interval* of $\mathbb{R}$ to be a subset $I \subseteq \mathbb{R}$ such that

   $$x < z < y \text{ and } x, y \in I \implies z \in I.$$

   (See optional question #1)
   Prove that if $E \subseteq \mathbb{R}$ is connected, then E is an interval.

   Solution: Take $x, y \in E$ and $z \in \mathbb{R}$ with $x < z < y$. To check E is an interval, we need to check that $z \in E$. Assume not. But then $U = (-\infty, z)$ and $V = (z, \infty)$ are disjoint opens covering E. Moreover, neither U nor V is empty as $x \in U$ and $y \in V$. Thus E is not connected which is a contradiction. Hence $z \in E$ and E is an interval.

4. A point P of a topological space X is called a *cut point*, if X is connected, but $X - \{P\}$ is not connected. Prove that the property of having a cut point is a topological property. Is it true that having exactly n cut points is a topological property?

   Solution: Let $f : X \to Y$ be a homeomorphism. Assume X is connected and that $P \in X$ is a cut point. We will show that $f(P)$ is then a cut point of Y. First note that Y is connected since being connected is a topological property. We now check that $Y - \{f(P)\}$ is not connected. First note that we know that $X - \{P\}$ is not connected since P is cut-point. Now our homeomorphism f restricts to a map $g : X - \{P\} \to Y - \{f(P)\}$. Note that g is again a homeomorphism since it is still bijective, continuous with continuous inverse. We then deduce that $Y - \{f(P)\}$ is not connected (again since being connected is a topological property). Thus $f(P)$ is a cut point of Y.

   Having exactly n cut points is also a topological property. The above argument shows that P is a cut point of X if and only if $f(P)$ is a cut point of Y. So $P_1, \ldots, P_n$ are the exact set of cut points of X if and only if $f(P_1), \ldots, f(P_n)$ are the exact set of cut points of Y.

5. Prove that $(0, 1)$ is not homeomorphic to $[0, 1)$.
   Hint: consider the spaces cut points.

   Solution: Every point of $(0, 1)$ is a cut point as $(0, z) \cup (z, 1)$ is not connected. However, 0 is not a cut point of $[0, 1)$ as $(0, 1)$ is connected. Any homeomorphism of $[0, 1)$ to $(0, 1)$ would then take 0 to a non-cut point of $(0, 1)$ which is impossible.

6. Group the letters of the alphabet by homeomorphism. That is, break the alphabet into groups of letters such that within each group all letters are homeomorphic and no letters in different groups are homeomorphic. (Let's all agree to use the fonts in the following website:

   `http://www.graffitiknowhow.com/wp-content/uploads/2014/06/sans-serif-alphabet1.jpg`)

   Do the best you can to justify your answers.

   Solution: This question sadly is quite ambiguous especially with our discussion in class in mind. The website of fonts that I directed you to has letters with definite "thickness". In my mind, letters were locally 1-dimensional, but these letters are not. If the letters were 1-dimensional, then the letter's would have cut points. But these thick letters don't have any cut points!

   In any case, let me at least write down my thoughts for these thick letters. I think they break down into 3 classes just depending on how many "holes" they have. So one class is:

   $$C, E, F, G, H, I, J, K, L, M, N, S, T, U, V, W, X, Y, Z.$$

   The thick versions of all of these letters can be deformed into a filled in square. Next we have

   $$A, D, O, P, Q, R$$

   which can all be stretched out into an annulus. Lastly, in a class by itself, we have

   $$B.$$

   We will soon see that these spaces all have different fundamental groups which is a topological invariant.

Optional questions:

1. Prove that $I \subseteq \mathbb{R}$ is an interval if and only if

   $$I = \varnothing, (a, b), [a, b), (a, b], [a, b], (-\infty, b), (-\infty, b], (a, \infty), [a, \infty) \text{ or } \mathbb{R}.$$