Topology – MA 564 – Spring 2015 – Midterm Solutions

Provide all answers in the blue book provided. Answer clearly and completely to receive full credit.

Notation: In all of the following questions, (X, d) and (Y, s) are metric spaces.

(8 points) Definitions (answer 1 of the following 2 questions)

1. Let A be a subset of X, and let $x \in X$. Define what it means for x *to be adherent to* A.

   Solution: x is adherent to $A \iff$ for every $r > 0$, $B_r(x) \cap A \neq \varnothing$.

2. Let $\{x_n\}$ be a sequence in X and let $x \in X$. Define what it means for $\{x_n\}$ *to converge to* x.

   Solution: $\{x_n\}$ to converge to $x \iff$ for every $\varepsilon > 0$, there exists $N \in \mathbb{R}$ such that if $n \geq N$, then $d(x_n, x) < \varepsilon$.

(24 points) Proofs (answer 2 of the following 3 questions)

1. Let $f : X \to Y$ be a function and let $C, D \subseteq Y$. Prove that $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.

   Solution:

$$\begin{aligned} x \in f^{-1}(C \cup D) &\iff f(x) \in C \cup D \iff f(x) \in C \text{ or } f(x) \in D \\ &\iff x \in f^{-1}(C) \text{ or } x \in f^{-1}(D) \iff x \in f^{-1}(C) \cup f^{-1}(D) \end{aligned}$$

2. Let $A \subseteq X$ and let $\{x_n\}$ be a sequence with each $x_n \in A$. If $\{x_n\}$ converges to x, prove that x is adherent to A.

   Solution: Take $r > 0$. Then there exists $N \in \mathbb{R}$ such that for $n \geq N$ we have $x_n \in B_r(x)$. But then $x_n \in A \cap B_r(x)$ and we deduce that x is adherent to A.

3. For $x \in X$, prove that $\displaystyle\bigcap_{n=1}^{\infty} B_{\frac{1}{n}}(x) = \{x\}$.

   Solution: Take $z \in \bigcap_{n=1}^{\infty} B_{\frac{1}{n}}(x) = \{x\}$. Then $z \in B_{\frac{1}{n}}(x)$ for all $n \geq 1$. Thus, $d(z, x) \leq \frac{1}{n}$ for all $n \geq 1$ and hence $d(z, x) = 0$. By the first axiom of a metric space this implies $z = x$ as desired.

(48 points) True/False — for each of the following questions answer True or False. If you answer False, you must justify your answer.

1. For any metric space (X, d) and $x \in X$, the singleton set $\{x\}$ is closed.

   Solution: TRUE

2. The only subsets of $\mathbb{R}$ which are both open and closed under the discrete metric are $\varnothing$ and $\mathbb{R}$.

   Solution: False. Every subset of a discrete metric space is open and closed. So, in particular, $\{0\}$ is both open and closed.

3. The only subsets of $\mathbb{R}$ which are both open and closed under the standard metric are $\varnothing$ and $\mathbb{R}$.

   Solution: TRUE

4. A subset $U \subseteq \mathbb{R}^2$ is open under the standard metric if and only if $U \subseteq \mathbb{R}^2$ is open under the taxi-cab metric.

   Solution: TRUE

5. Let $X = [0,1] \cup [2,3]$ endowed with the standard metric. Then $[0,1]$ is an open subset of X.

   Solution: TRUE

6. For $A \subseteq X$, we have $\operatorname{int}(\overline{A}) = A$.

   Solution: FALSE. Take $X = \mathbb{R}$ with the usual metric and take $A = \{0\}$. Then $\operatorname{int}(\overline{A}) = \operatorname{int}(\overline{\{0\}}) = \operatorname{int}(\{0\}) = \varnothing \neq \{0\} = A$.

7. If $A, B \subseteq X$ with $\overline{A} = \overline{B}$ and $\operatorname{int} A = \operatorname{int} B$, then $A = B$.

   Solution: FALSE. Take $X = \mathbb{R}$, $A = (0,1)$ and $B = [0,1]$. Then $\overline{A} = [0,1] = \overline{B}$ and $\operatorname{int}(A) = (0,1) = \operatorname{int}(B)$ but $A \neq B$.

8. For $A \subseteq X$, if $\operatorname{int}(A) = \overline{A}$, then A is both closed and open.

   Solution: TRUE

9. If $f : X \to Y$, $A, B \subseteq X$, then $f(A \cap B) = f(A) \cap f(B)$.

   Solution: FALSE. Consider $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Let $A = \{-1\}$ and $B = \{1\}$. Then $f(A \cap B) = \varnothing$ while $f(A) \cap f(B) = \{1\}$.

10. If $f : X \to Y$ is continuous and $U \subseteq X$ is open, then $f(U)$ is open.

    Solution: FALSE. Consider $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Then $f((-1,1)) = [0,1)$. Note that $(-1,1)$ is open while $[0,1)$ is not.

11. If $f : X \to Y$ is continuous and $\{x_n\} \to x$ in X, then $\{f(x_n)\} \to f(x)$ in Y.

    Solution: TRUE

12. If $f : X \to Y$ is continuous at $x \in X$, and $U \subseteq Y$ is an open set containing $f(x)$, then $f^{-1}(U)$ is open in X.

    Solution: FALSE. Consider $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$ which is continuous at $x = 1$. Take $U = (1/2, 3/2)$ which contains $1 = f(1)$. However, $f^{-1}(U) = [0, \infty)$ which is not open.