Algebraic Number Theory
MA844 (aka MA743)
Spring 2014
HW #5

(1) Recall the p-adic valuation $\mathrm{ord}_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$. (Here we are extending its definition so that $\mathrm{ord}_p(0) = \infty$.) Prove:
  (a) $\mathrm{ord}_p(a + b) \geq \min\{\mathrm{ord}_p(a), \mathrm{ord}_p(b)\}$;
  (b) if $\mathrm{ord}_p(a) \neq \mathrm{ord}_p(b)$, then $\mathrm{ord}_p(a + b) = \min\{\mathrm{ord}_p(a), \mathrm{ord}_p(b)\}$.

(2) Recall the p-adic absolute value $|\cdot|_p : \mathbb{Q} \to \mathbb{R}^{\geq 0}$.
  (a) Let $D(a, r) := \{x \in \mathbb{Q} \ : \ |x - a|_p < r\}$ denote the "open" disc in $\mathbb{Q}$. Show that $D(a, r)$ is open. Show that $D(a, r)$ is closed.
  (b) Let $\overline{D}(a, r) := \{x \in \mathbb{Q} \ : \ |x - a|_p \leq r\}$ denote the "closed" disc in $\mathbb{Q}$. Show that $\overline{D}(a, r)$ is closed. Show that $\overline{D}(a, r)$ is open.
  (c) Show that "every triangle is isoceles" in $\mathbb{Q}$ under $|\cdot|_p$.
  (d) Show that any point in $D(a, r)$ is a center. That is, if $b \in D(a, r)$ then $D(a, r) = D(b, r)$.

(3) Neukrich, Chapter 2, section 1: 4,5

(4) Neukrich, Chapter 2, section 2: 5,6

(5) Prove $\mathbb{Q}_p(\zeta_p) \cong \mathbb{Q}_p(p^{1/(p-1)})$.

(6) How does the 7-th cyclotomic polynomial factor in $\mathbb{Q}_{11}$? Explain.

(7) Show that $\mathbb{Q}_p^{\mathrm{un}}$ is not complete under the p-adic norm.

(8) Let $K/\mathbb{Q}$ be a number field. Any embedding from K to $\overline{\mathbb{Q}}_p$ induces a non-archimedian absolute value on K (by restriction).
  (a) Show that this absolute value on K is equivalent to the absolute value associated to some (unique) prime ideal sitting over p.
  (b) Is this association of embeddings of K into $\overline{\mathbb{Q}}_p$ to primes of K over p bijective?

(9) Show that maximal abelian extension of $\mathbb{Q}(i)$ is larger than the $\mathbb{Q}^{ab}(i)$. (Hint: find an abelian extension of $\mathbb{Q}(i)$ which is not abelian over $\mathbb{Q}$.) Can you generalize your argument to any quadratic extension? How about any number field? How about replacing $\mathbb{Q}$ by $\mathbb{Q}_p$?