ON COSTELLO’S CONSTRUCTION OF THE WITTEN GENUS: 
$L_\infty$ SPACES AND DG-MANIFOLDS

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1. $L_\infty$ SPACES

An $L_\infty$ space is a ringed space with a structure sheaf a sheaf $L_\infty$ algebras, where an $L_\infty$ algebra is the homotopical enhancement of a differential graded Lie algebra.

1.1. $L_\infty$ algebras. Let $A$ a differential commutative graded algebra and $I \subset A$ a nilpotent ideal. Let $A^\#$ denote the underlying graded algebra i.e. we forget the differential.

**Definition 1.1.** A curved $L_\infty$ algebra over $A$ consists of a locally free finitely generated graded $A^\#$-module $V$, together with a cohomological degree 1 and square zero derivation $d : \widehat{\text{Sym}}(V^*[−1]) → \widehat{\text{Sym}}(V^*[−1])$

where $V^*$ is the $A^\#$-linear dual and the completed symmetric algebra is also over $A^\#$.

There are two additional requirements on the derivation $d$:

1. The derivation $d$ makes $\widehat{\text{Sym}}(V^*[−1])$ into a dga over the dga $A$;
2. Reduced modulo the nilpotent ideal $I \subset A$, the derivation $d$ preserves the ideal in $\widehat{\text{Sym}}(V^*[−1])$ generated by $V$.

Note that our dualizing convention is such that $V^*[−1] = V[1]^*$.

We can decompose the derivation $d$ into its constituent pieces

$$d_n : V^*[−1] → \text{Sym}^n(V^*[−1])$$

and after dualizing and shifting we obtain maps

$$l_n : \Lambda^n V[2−n] → V.$$ 

The maps $\{l_n\}$ satisfy higher Jacobi relations. In particular, if $l_n = 0$ for all $n ≠ 2$, then $V$ is just a graded Lie algebra. Similarly, if $l_n = 0$ for all $n ≠ 1,2$, then $V$ is a differential graded Lie algebra. If $l_n = 0$ for $n ≠ 1,2,3$ then $l_3$ is a contracting homotopy for the Jacobi relation, i.e.

$$(−1)^{|x||z|}l_2(l_2(x,y),z) + (−1)^{|y||z|}l_2(l_2(z,x),y) + (−1)^{|x||y|}l_2(l_2(y,z),x) =$$

$$(−1)^{|x||z|+1}(l_1l_3(x,y,z) + l_3(l_1x,y,z) + (−1)^{|x|}l_3(x,l_1(y),z) + (−1)^{|x|+|y|}l_3(x,y,l_1z)).$$

If $V$ is a $L_\infty$ algebra over $A$, then $C^*(V)$ will denote the differential graded $A$-algebra $\widehat{\text{Sym}}(V^*[−1])$. Our convention will be that $V$ is concentrated in non-negative degrees, so that $C^{>0}(V)$ is a (maximal) ideal of $C^*(V)$.
1.2. \( L_\infty \) spaces. Let \( X \) be a manifold and consider the nilpotent ideal \( \Omega_X^0 \subset \Omega_X^\ast \).

**Definition 1.2.** An \( L_\infty \) space is a manifold \( X \) equipped with a sheaf \( \mathfrak{g} \) of \( L_\infty \) algebras over \( \Omega_X^\ast \) which is locally free of finite total rank (as graded \( \Omega_X^\ast \)-modules).

**Definition 1.3.** Given an \( L_\infty \) space \( (X, \mathfrak{g}) \), the reduced structure sheaf \( \mathfrak{g}_{\text{red}} \) is defined by

\[
\mathfrak{g}_{\text{red}} = \mathfrak{g} / \Omega_X^0.
\]

One should think of the reduced structure sheaf as something like the dual to the cotangent complex and hence a measure of the “niceness” of the \( L_\infty \) space \( (X, \mathfrak{g}) \).

**Proposition 1.4.** Given an \( L_\infty \) space \( (X, \mathfrak{g}) \), the reduced structure sheaf \( \mathfrak{g}_{\text{red}} \) has no curving i.e. \( \ell_1^2 = 0 \).

**Proof.** From the \( L_\infty \) relations we know that \( \ell_1^2 = \ell_0 \). Now \( \ell_0 : C \rightarrow V \) i.e. \( \ell_0 \) is just an element of \( V \) which is dual to the map \( d_0 : V^\vee [-1] \rightarrow C \). The condition that reduced modulo the nilpotent ideal \( I \) the derivation \( d \) preserves the ideal generated by \( V \) implies that \( \ell_0 \in V \otimes_A I \). Therefore reduced modulo \( I, \ell_0 = 0 \). \( \square \)

1.3. **Morphisms of \( L_\infty \) spaces.** A map \( \alpha : \mathfrak{g} \rightarrow \mathfrak{h} \) of \( L_\infty \) algebras given by a sequence of linear maps

\[ \text{Sym}^n(\mathfrak{g}[1]) \rightarrow \mathfrak{h} \]

of degree 1 satisfying certain quadratic identitites. If \( \mathfrak{h} \) is finite dimensional, then the map \( \alpha \) is exactly a map of differential graded algebras

\[ C^*(\mathfrak{h}) \rightarrow C^*(\mathfrak{g}) \]

which takes the maximal ideal \( C^{>0}(\mathfrak{h}) \) to the maximal ideal \( C^{>0}(\mathfrak{g}) \). Alternatively, we can view \( \alpha \) as an element \( \alpha \in C^*(\mathfrak{g}) \otimes \mathfrak{h} \) satisfying the Maurer-Cartan equation

\[ d\alpha + \sum_{n>1} \frac{1}{n!} l_n(\alpha, \ldots, \alpha) = 0 \]

and which vanishes modulo the maximal ideal \( C^{>0}(\mathfrak{g}) \).\footnote{The idea is that we have the following subspace}

\[ \text{hom}_{\text{dgAlg}}(C^*(\mathfrak{h}), C^*(\mathfrak{g})) \subset \text{hom}_{grAlg}(C^*(\mathfrak{h}), C^*(\mathfrak{g})) \cong \text{hom}_{grVect}(\mathfrak{h}^\vee, C^*(\mathfrak{g})) \cong (C^*(\mathfrak{g}) \otimes \mathfrak{h})_1. \]

That the map determined by \( \alpha \in (C^*(\mathfrak{g}) \otimes \mathfrak{h})_1 \) respects the differential is exactly the Maurer-Cartan equation.
Let \((X, g_X)\) be an \(L_\infty\) space and \(Y\) is a smooth manifold. Given a smooth map \(\phi : Y \to X\) we have the pull back \(L_\infty\) algebra over \(\Omega_Y^*\) given by

\[
\phi^* g_X \overset{d{f}}{=} \phi^{-1} g_X \otimes \Omega_Y^*. 
\]

Here \(\phi^{-1} g_X\) denotes the sheaf pull back.

**Definition 1.5.** Let \((X, g_X)\) and \((Y, g_Y)\) be \(L_\infty\) spaces, then a morphism

\[
\phi : (Y, g_Y) \to (X, g_X)
\]

is given by a smooth map \(\phi : Y \to X\) and a map of curved \(L_\infty\) algebras over \(\Omega_Y^*\)

\[
g_Y \to \phi^* g_X.
\]

We also have the notion of equivalence for \(L_\infty\) algebras as cochain homotopy equivalence of the reduced algebras which leads to a definition of equivalence on the space level.

**Definition 1.6.** An \(L_\infty\) map \(\phi : (Y, g_Y) \to (X, g_X)\) is an equivalence of \(L_\infty\) spaces if the underlying map \(Y \to X\) is a diffeomorphism and if the map of curved \(L_\infty\) algebras \(g_Y \to \phi^* g_X\) is an equivalence i.e. the map

\[
(g_Y)_{red} \to (\phi^* g_X)_{red}
\]

is a cochain homotopy equivalence of sheaves \(C^\infty_Y\) modules.

This notion of equivalence is quite strong. If \(g\) and \(h\) are equivalent as curved \(L_\infty\) algebras over \(\Omega_Y^*\), then \(C^*(g)\) and \(C^*(h)\) are homotopy equivalent, but the converse is not necessarily true. Note that \(C^*(g)\) (and similarly for \(C^*(h)\)) are filtered by powers of the ideal generated by \(\Omega_Y^{>0}\) and \(g^\lor\), the associated graded is \(\text{Sym}(g^\lor_{red}[-1])\). The definition of equivalence implies that we have an equivalence at the first page of the associated spectral sequences². One reason why this stronger definition is desirable is if we consider \(L_\infty\) spaces with underlying manifold a point, then there are \(L_\infty\) algebras (even just Lie algebras) that have quasi-isomorphic Chevalley-Eilenberg complexes yet that are quite different; for instance, it is well known that \(H^*(\mathfrak{sl}_2(\mathbb{C}))\) is an exterior algebra on one generator in each degree \(3n\) for \(n \geq 0\), but the rank 1 free Abelian lie algebra concentrated in degree -2 has the same cohomology.

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²Recall that for a filtered complex \((F^*, C^*, d)\) with \(F^{p+1}C^* \subseteq F^pC^*\) we have an associated (cohomological) spectral sequence with \(E_0\)-page given by

\[
E_0^{p,q} = \frac{F^pC^{p+q}}{F^{p+1}C^{p+q}}, \quad d_0 = d : C^{p+q} \to C^{p+q+1}.
\]

If the spectral sequence converges (e.g. for bounded filtered complexes) then we have

\[
E_\infty^{p,q} = \text{Gr}_p H^{p+q}(C^*).
\]
Remark 1.7. Note that the category of \( L_\infty \) spaces can be simplicially enriched. The \( n \)-simplices of the set of maps \((Y,g_Y)\) to \((X,g_X)\) are smooth maps \( \phi : Y \to X \) and a map of curved \( L_\infty \) algebras over \( \Omega_Y \)

\[
g_Y \to \phi^* g_X \otimes \Omega^*_\Delta
\]

where the right hand side makes sense as \( L_\infty \) algebras are tensored over cdgas. One advantage of this perspective is that it allows us to define families of \( L_\infty \) structures and a natural notion of homotopy. It is non trivial, yet true (as shown in [?]) that the simplicial structure is compatible with the definition of equivalence in \( L_\infty \) spaces. i.e if \( \phi : (Y,g_Y) \to (X,g_X) \) is an equivalence then for any other \( L_\infty \) space \((Z,g_Z)\) the induced maps of of simplicial sets

\[
\text{Maps}((Z,g_Z),(Y,g_Y)) \to \text{Maps}((Z,g_Z),(X,g_X))
\]

\[
\text{Maps}((X,g_X),(Z,g_Z)) \to \text{Maps}((Y,g_Y),(Z,g_Z))
\]

are weak homotopy equivalences.

2. GEOMETRIC CONSTRUCTIONS ON \( L_\infty \) SPACES

Definition 2.1. Let \((X,g)\) be an \( L_\infty \) space.

- A vector bundle \( V \) on \((X,g)\) is a locally free sheaf of \( \Omega_X \) modules such that \( V \oplus g \) has the structure of a curved \( L_\infty \) algebra over \( \Omega_X \) satisfying
  - The maps \( g \hookrightarrow V \oplus g \) and \( V \oplus g \to g \) are maps of \( L_\infty \) algebras;
  - The Taylor coefficients \( l_n \) vanish on tensors containing two or more sections of \( V \).
- The sheaf of sections of \( V \) is given by \( C^*(g,V[1]) \) as a sheaf of dg modules over \( C^*(g) \).

The \( L_\infty \) space \((X,V \oplus g)\) is the total space of the vector bundle given by \( V[1] \) formally completed along the zero section.

2.1. (Co)Tangent bundle. Let \( V \) be a vector space (finite dimensional or topological) which we can think of as a dg-manifold with underlying manifold a point. We define functions on \( V \) by the (completed) symmetric algebra of the dual. Now functions on the tangent bundle \( T(V) \) are given by

\[
\mathcal{O}(TV) = \mathcal{O}_V \otimes_{\mathcal{O}_V} \text{Sym}_{\mathcal{O}_V}(\text{Der}(\mathcal{O}_V)^\vee).
\]

Hence, the tangent bundle \( T(X,g) \) is given by the \( g \) module \( g[1] \). Vector fields are sections of the tangent bundle and hence as a sheaf are \( C^*(g,g[1]) \). Another way to see this is as follows. Consider any graded vector bundle \( E \to X \), then for any \( U \subset X \) open we have an identification of vector spaces

\[
\text{Der}(\mathcal{O}(E(U))) \cong \mathcal{O}(E(U)) \otimes E(U).
\]

This equivalence follows from the fact that any such derivation is determined by its value on the generators and hence is determined by a map

\[
E(U)^\vee \to \mathcal{O}(E(U)).
\]
We define the cotangent bundle $T^*(X, g)$ to be the dual module to the tangent bundle i.e. $g^\vee[-1]$. A $k$-form on $(X, g)$ is a section of the $k$th exterior power of the cotangent bundle, where

$$\Lambda^k T^*(X, g) = \Lambda^k (g^\vee[-1]) = \text{Sym}^k (g^\vee)[-k].$$

So a $k$-form is a section of the sheaf $C^*(g, \text{Sym}^k (g^\vee)[-k])$.

The total space of the tangent bundle is given by $T(X, g) = (X, g \oplus g)$, while the total space of the cotangent bundle is given by $T^*(X, g) = (X, g \oplus g^\vee[-2])$.

We also have the shifted version of the tangent and cotangent bundles. Of note, we have

$$T^*[-1](X, g) = (X, g \oplus g^\vee[-3]) \text{ and } T[-1](X, g) = (X, g[e])$$

where $e$ is a square zero parameter of degree 1.

### 3. DG RINGED MANIFOLDS

**Definition 3.1.** A dg ringed manifold is a manifold $M$, together with a sheaf $\mathcal{A}$ of differential graded unital $\Omega^*_M$-algebras such that

1. As a sheaf of $\Omega^0_M$-algebras, $\mathcal{A}$ is locally free of finite total rank;
2. $\mathcal{A}$ is equipped with a map of sheaves of $\Omega^*_M$-algebras $\mathcal{A} \to C^\infty_M$, the kernel of this map must be a sheaf of nilpotent ideals;
3. For sufficiently small open subsets $U$ of $M$, the cohomology of $\mathcal{A}(U)$ must be concentrated in non-positive degrees.

Note that the conditions imply that the $\Omega^0_M$-module $\mathcal{A}$ is given by the sections of a graded vector bundle of finite total rank on $M$.

**Example 3.2.**

1. Let $M$ be any manifold and let $\mathcal{A} = \Omega^*_M$ equipped with the de Rham differential. The resulting dg ringed space is denoted $M_{dR}$.
2. Let $M$ be any manifold and let $\mathcal{A} = C^\infty_M$. We denote the resulting dg ringed space by $M$.
3. Let $M$ be a complex manifold. There is a complex (i.e. we work over $\Omega^*_M \otimes_R \mathbb{C}$) dg ringed space $M_{\overline{\partial}}$ with $\mathcal{A} = \Omega^0_{\overline{\partial}}(M)$ and differential $\overline{\partial}$.
4. If $(X, g)$ is an $L_\infty$ space, then $(X, C^*(g))$ is a pro-dg ringed manifold.

**Definition 3.3.** A map of dg ringed manifolds $(M, \mathcal{A}) \to (N, \mathcal{B})$ is a smooth map $f : M \to N$, together with a map of sheaves of dg $f^{-1}\Omega^*_N$-algebras $f^{-1}\mathcal{B} \to \mathcal{A}$, such that the
If \((M, \mathcal{A})\) is a dg ringed manifold, then the sheaf \(\mathcal{A}\) is filtered by the powers of the nilpotent ideal \(\mathcal{I} \subset \mathcal{A}\) which is the kernel of the map \(\mathcal{A} \to \mathcal{C}^\infty_M\). Let \(\text{Gr} \mathcal{A}\) denote the associated graded sheaf of dg algebras over \(\Omega^*_{M}\).

**Definition 3.4.** A map \((M, \mathcal{A}) \to (N, \mathcal{B})\) of dg ringed manifolds is an equivalence, if the map of smooth manifolds \(M \to N\) is a diffeomorphism, and the induced map of sheaves \(\text{Gr} \mathcal{A} \to \text{Gr} \mathcal{B}\)
is a quasi-isomorphism.