ON COSTELLO’S CONSTRUCTION OF THE WITTEN GENUS:  
MAPPING SPACES  
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Goal: Study the space of holomorphic maps \( E \rightarrow X \), for \( E \) an elliptic curve and \( X \) a complex manifold.

Goal’: Study the space of holomorphic maps \( E \rightarrow X \) which are infinitesimally close to a constant map.

Setting: The space of such maps is a simplicial presheaf (actually a derived space) on the category of dg ringed manifolds, i.e.

\[
\text{Map}_{\text{const}}(E, X) : \text{dgMan} \rightarrow s\text{Set}.
\]

Note this notation is ad hoc and will be cleared up below.

Main Theorem: The simplicial presheaf \( \text{Map}_{\text{const}}(E, X) \) is representable.

1. DERIVED GEOMETRY WITH \( L_\infty \) ALGEBRAS

We are interested in studying formal derived moduli problems, as an orienting remark recall that particularly nice simplicial sets are those that are Kan complexes and that the nerve of a category \( C \) is Kan complex if and only if \( C \) is a groupoid. Consider the following progression.

- Schemes: Functors from commutative algebras to sets;
- Stacks: Functors from commutative algebras to groupoids;
- Higher Stacks: Functors from commutative algebras to simplicial sets;
- Derived Stacks: Functors from dg commutative algebras to simplicial sets

Further, the adjective formal restricts the domain category to (local) Artinian algebras.\(^1\)

We make the following definition.

**Definition 1.1.** A formal derived moduli problem is a functor

\[
F : \text{Artinian cdga} \rightarrow s\text{Set}
\]

such that

1. \( F(C) \) is contractible (unique point);
2. \( F \) takes surjective maps of dg Artinian rings to fibrations of simplicial sets;

\(^1\)There doesn’t seem to be an agreed upon definition of an Artinian cdga. Costello calls a cdga \( R \) Artinian if it is concentrated in non-positive degrees, bounded below, and each graded component is finite dimensional. Further, \( R \) must have a unique maximal differential idea \( m \) such that \( R/m = \mathbb{C} \), with \( m \) nilpotent.
If $A, B, C$ are dg Artinian rings, and $B \to A, C \to A$ are surjective maps, then we require that the natural map

$$F(B \times_A C) \to F(B) \times_{F(A)} F(C)$$

is a weak homotopy equivalence.

Every $L_\infty$ algebra gives rise to a formal (derived) moduli problem. Indeed, let $\mathfrak{g}$ be an $L_\infty$ algebra, we define a functor $MC_{\mathfrak{g}} : \text{Artinian cdga} \to \text{sSet}$ whose set of $n$-simplices associated to $(R, m)$ is given by

$$MC_{\mathfrak{g}}(R)[n] = \left\{ \alpha \in (\mathfrak{g} \otimes m \otimes \Omega^*(\Delta^n))_1 : \sum_{n!} \frac{1}{n!} l_n(\alpha^\otimes n) = 0 \right\}.$$

Note that $L_\infty$ algebras are tensored over cdgas.

**Example 1.2.** Let $\mathfrak{g} = V[-1]$ with $V$ a finite dimensional vector space (all brackets are trivial). If $(R, m)$ is an ordinary Artinian ring (not dg), then

$$MC_{\mathfrak{g}}(R)[n] = V \otimes m,$$

for all $n$. Hence, $MC_{\mathfrak{g}}(R)$ is a constant simplicial set and represents

$$V \otimes m = \{ \text{pointed maps : Spec } R \to V \}.$$

**Example 1.3.** Let $\mathfrak{g} = C[-1]$ and $(R, m)$ a dg Artin ring. Then

$$MC_{\mathfrak{g}}(R)[n] = \left\{ \alpha \in (m \otimes \Omega^*(\Delta^n))_0 : d\alpha = 0 \right\}$$

which is the Dold-Kan simplicial set for the cochain complex $m$; it represents pointed maps of derived schemes

$$\text{Spec } R \to \mathbb{A}^1.$$

Note that this functor is also represented by the pro-Artinian algebra $C[[t]]$, in the sense that mapping space $C[[t]] \to (R, m)$ can be identified with $MC_{\mathfrak{g}}(R)$.

**Example 1.4.** Let $G$ be an algebraic group with Lie algebra $\mathfrak{g}$. If $R$ is an Artinian ring (not dg), then

$$MC_{\mathfrak{g}}(R) \simeq BG(R),$$

where $BG$ is the classifying space of $G$.

**Example 1.5.** Consider the elliptic $L_\infty$ algebra $\mathcal{L}$ describing free scalar field theory on a Riemannian manifold $M$ with associated Laplacian $D$, where $\mathcal{L}$ is concentrated in degrees 1 and 2 with only a differential and no higher brackets:

$$\mathcal{L} = C_M^\infty \xrightarrow{D} C_M^\infty.$$

To begin let us analyze the associated Maurer-Cartan functor evaluated on an ordinary (non dg) Artinian ring $R$ with maximal ideal $m$. Then we have

$$MC_{\mathcal{L}}(R, m)[0] = \{ \phi \in C_M^\infty \otimes m : D\phi = 0 \}.$$
All higher simplices are constant. Indeed, if \( \phi \in L \otimes m \otimes \Omega^* (\Delta^n) \) is a closed element of degree 1, then

\[
\phi \in C^\infty_M \otimes m \otimes \Omega^0 (\Delta^n) \quad \text{and} \quad D \phi = 0 = d_{dR} \phi,
\]

where \( d_{dR} \) is the de Rham differential on \( \Omega^* (\Delta^n) \).

Now consider the Maurer-Cartan simplicial set associated to a dg Artinian ring \((R, m, d_R)\). We have

\[
MC_L (R, m)[0] = \{ \phi \in C^\infty_M \otimes m_0, \psi \in C^\infty_M \otimes m_{-1} : D \phi = d_{dR} \psi \},
\]

so the zero-simplices are identified with \( R \)-valued smooth functions \( \phi \) on \( M \) which are harmonic up to a homotopy given by \( \psi \) and which vanish modulo the maximal ideal \( m \).

A 1-simplex is consists of four terms:

\[
\begin{align*}
\phi_0 (t) & \in C^\infty_M \otimes m_0 \otimes \Omega^0 ([0, 1]) \\
\phi_1 (t) dt & \in C^\infty_M \otimes m_{-1} \otimes \Omega^1 ([0, 1]) \\
\psi_0 (t) & \in C^\infty_M \otimes m_{-1} \otimes \Omega^0 ([0, 1]) \\
\phi_1 (t) dt & \in C^\infty_M \otimes m_{-2} \otimes \Omega^1 ([0, 1]),
\end{align*}
\]

that are closed in the following sense

\[
\begin{align*}
D \phi_0 (t) &= d_R \psi_0 (t) \\
\frac{d}{dt} \phi_0 (t) &= d_R \phi_1 (t) \\
D \phi_1 (t) + \frac{d}{dt} \psi_0 (t) &= d_R \psi_1 (t).
\end{align*}
\]

That is, \( \phi_0 (t) \) is a family of \( R \)-valued smooth functions on \( M \), which are harmonic up to a homotopy specified by \( \psi_0 (t) \). Further, \( \phi_0 (t) \) is independent of \( t \), up to a homotopy specified by \( \phi_1 (t) \). Finally, we have a coherence condition between our two homotopies.

### 1.1. A global Maurer-Cartan functor

We now globalize the previous construction. That is, to an \( L_\infty \) space \((X, g)\) we associate a derived space i.e. a functor from dg ringed manifolds to simplicial sets which preserves equivalences and satisfy Čech descent.

**Definition 1.6.** Let \((X, g)\) be an \( L_\infty \) space. Define a simplicial pre sheaf \( MC_{(X, g)} \) on the category of dg ringed manifolds by declaring \( MC_{(X, g)} (M, \mathcal{A}) \) to be the simplicial set whose \( n \)-simplices consist of smooth maps \( f : M \to X \), together with a Maurer-Cartan element

\[
\alpha \in f^* g \otimes \Omega^*_M \mathcal{A} \otimes_R \Omega^* (\Delta^n)
\]

which vanishes modulo the nilpotent ideal \( \mathcal{I} \subset \mathcal{A} \).

Recall that the ideal \( \mathcal{I} \subset \mathcal{A} \) is the kernel of the map of sheaves \( \mathcal{A} \to C^\infty_M \). To give a Maurer-Cartan element as above (really a 0-simplex) is then the same as to give a map of (pro)-\( \Omega^*_M \)-algebras

\[
C^* (f^* g) \to \mathcal{A}
\]
such that the following diagram commutes

\[
\begin{array}{ccc}
C^\ast (f^\ast \mathfrak{g}) & \longrightarrow & \mathfrak{A} \\
\downarrow & & \downarrow \\
C^\infty_M & \longrightarrow & 
\end{array}
\]

1.2. **The Maurer-Cartan functor associated to** \((X, \mathfrak{g}_X)\). Recall the \(L_\infty\) space \((X, \mathfrak{g}_X)\) which encodes the complex structure of \(X\), \(\mathfrak{g}_X \cong T_{X}^{1,0}[-1] \otimes \Omega^*_{X}\). Let \(M\) be a complex manifold and consider the dg ringed space \((M, \Omega^0_M)\).

**Proposition 1.7** (3.1.1 of Costello’s Witten Genus II). Let \(M\) and \(X\) be complex manifolds.

1. The simplicial set \(MC_{(X, \mathfrak{g}_X)}(M, \Omega^0_M)\) is discrete, that is, all higher simplices are constant.
2. Zero simplices of \(MC_{(X, \mathfrak{g}_X)}(M, \Omega^0_M)\) are in bijection with holomorphic maps \(M \to X\).

**Proof.** Fix a smooth map \(\phi : M \to X\) and observe that the nilpotent curved \(L_\infty\) algebra \(\Omega^0_M \otimes_{\Omega^0_M} \phi^* \mathfrak{g}_X\) is concentrated in cohomological degrees greater than 1. Hence, if a Maurer-Cartan exists, it is unique and the simplicial set of Maurer-cartan elements is discrete.

What remains is to consider the curving \(l_0\). If \(l_0\) vanishes, then we have a commutative diagram of \(\Omega^*_M\)-algebras

\[
\begin{array}{ccc}
C^\ast (\phi^* \mathfrak{g}_X) & \longrightarrow & \Omega^0_M \\
\downarrow & & \downarrow \\
C^\infty_M & \longrightarrow & 
\end{array}
\]

From which we deduce that the pullback of a holomorphic function on \(X\) is a holomorphic function on \(M\), so \(\phi\) is holomorphic.

Conversely, suppose \(\phi\) is holomorphic. Then, \(\phi\) induces a map of \(\phi^{-1} \Omega^*_X\)-algebras

\[
\phi^{-1} \Omega^*_X \to \Omega^0_M.
\]

There is a natural map of \(\Omega^*_X\)-algebras

\[
\Omega^*_X(J(\partial_X)) \cong \Omega^0_X \otimes_{\partial_X} \Omega^{*,0}(J(\partial_X)) \to \Omega^0_M.
\]

Since all these maps are compatible with the pullback of modules (it boils down to a base change argument), it then follows that there is a map of \(\Omega^*_M\)-algebras

\[
C^\ast (\phi^* \mathfrak{g}_X) \to \Omega^0_M,
\]

so as a consequence of satisfying the Maurer-Cartan equation, \(l_0\) must vanish. \(\square\)
2. DERIVED MAPPING SPACES

We are studying the space of maps from a dg ringed manifold \((M, \mathcal{A})\) to an \(L_\infty\) space \((X, g)\). One should think of this construction as constructing an internal hom on the category of generalized dg-manifolds (after embedding \(L_\infty\) spaces inside of dg-manifolds).

The data of a single map is a smooth map \(\phi : M \to X\) and a Maurer-Cartan element \(\alpha \in \phi^* g \otimes \Omega^*_M \mathcal{A}\) which vanishes modulo the nilpotent ideal \(\mathcal{I} \subset \mathcal{A}\). As we described above, the space of all maps is naturally a simplicial presheaf on the category of dg ringed manifolds which we denote \(\text{MC}((M, \mathcal{A}), (X, g))\); it associates to a dg ringed manifold \((N, B)\) the simplicial set of maps from the dg ringed manifold \((N \times M, B \boxtimes \mathcal{A})\) to the \(L_\infty\) space \((X, g)\) i.e. a smooth map and a Maurer-Cartan element. Now define the subsheaf \(\widehat{\text{MC}}((M, \mathcal{A}), (X, g)) \subset \text{MC}((M, \mathcal{A}), (X, g))\) to be the subset of maps where the map on the underlying manifolds is constant.

2.1. Representability. Under certain conditions, the functor \(\widehat{\text{MC}}((M, \mathcal{A}), (X, g))\) is representable by an \(L_\infty\) space.

Proposition 2.1 (5.0.1 of Costello’s Witten Genus II). Let \((M, \mathcal{A})\) be a dg ringed manifold with the property that, if \(F^i \mathcal{A}(M)\) denotes the filtration on \(\mathcal{A}(M)\) by the powers of the ideal \(\mathcal{I}(M)\), then the cohomology of each \(\text{Gr}^i \mathcal{A}(M)\) for \(i \geq 1\) is concentrated in degrees \(\geq 1\).

Let \((X, g)\) be an \(L_\infty\) space such that the cohomology of the sheaf of \(L_\infty\) algebras \(g_{\text{red}}\) is concentrated in degrees \(\geq 1\).

Then, the restricted Maurer-Cartan functor \(\widehat{\text{MC}}((M, \mathcal{A}), (X, g))(N, B)\) is equivalent to the functor represented by the \(L_\infty\) space \((X, g \otimes \mathcal{A}(M))\).

Combining this proposition with Proposition 1.7, we conclude the following.

Corollary 2.2. Let \(E\) be an elliptic curve and \(X\) a complex manifold. The mapping space of holomorphic maps \(E \to X\) infinitesimally close to a constant map is represented by the \(L_\infty\) space

\[
\left( X, \Omega^{0,*}(E) \otimes g_{X_\mathcal{I}} \right).
\]

Sketch of proof of the Proposition. It is sufficient to fix a smooth map \(\phi : N \to X\) and analyze the resulting simplicial sets. Unwinding definitions we find that an \(n\)-simplex of \(\widehat{\text{MC}}((M, \mathcal{A}), (X, g))(N, B)\) over \(\phi\) is given by a Maurer-Cartan element in

\[
\phi^* g \otimes \Omega^*_N B \otimes \mathcal{A}(M) \otimes \Omega^*(\Delta^n)
\]

which vanishes modulo the ideal

\[
\mathcal{I}_1[n] = (\mathcal{I} \otimes \mathcal{A}(M) \otimes \Omega^*(\Delta^n)) + (B \otimes \mathcal{I}(M) \otimes \Omega^*(\Delta^n)).
\]

\[\text{3}\]More precisely, \(\widehat{\text{MC}}((M, \mathcal{A}), (X, g))(N, B)\) is the sub-simplicial set consisting of those maps \(N \times M \to X\) that factor through the projection \(N \times M \to N\).
Similarly, an $n$-simplex of $\text{MC}_{(X, g \otimes \mathcal{A}(M))}(N, \mathcal{B})$ over $\phi$ is a Maurer-Cartan element of

$$\phi^* \mathfrak{g} \otimes \Omega^*_N \mathcal{B} \otimes_R \mathcal{A}(M) \otimes_R \Omega^*(\Delta^n)$$

which vanishes modulo the ideal

$$\mathcal{I}_2[n] = \mathcal{I}_B \otimes \mathcal{A}(M) \otimes \Omega^*(\Delta^n).$$

The resulting simplicial sets are very similar, except that the second vanishing condition is stronger as $\mathcal{I}_2[n] \subset \mathcal{I}_1[n]$. Therefore, there is a natural transformation

$$\text{MC}_{(X, g \otimes \mathcal{A}(M))}(N, \mathcal{B}) \to \tilde{\text{MC}}((M, \mathcal{A}, (X, g))(N, \mathcal{B})).$$

It remains to prove that this natural transformation induces a weak equivalence upon evaluation for an arbitrary $(N, \mathcal{B})$.

The relevant simplicial sets (after fixing $\phi$) are those Maurer-Cartan simplicial sets associated to the nilpotent curved $L_\infty$ algebra

$$\mathfrak{g}_{i, \phi} = \Gamma(N, \phi^* \mathfrak{g} \otimes \Omega^*_N \mathcal{I}_i)$$

for $i = 1, 2$ and with $\mathcal{I}_i \equiv \mathcal{I}_i[0]$. These nilpotent $L_\infty$ algebras are equipped with finite bifiltrations induced by the filtrations on $\mathcal{A}(M)$ and $\mathcal{B}(N)$ by the powers of the ideals $\mathcal{I}_A(M)$ and $\mathcal{I}_B(N)$. By exhibiting a map which induces an isomorphism on the cohomology of the associated gradeds and proceeding by induction with respect to the filtration, one can deduce the proposition. □