

A contour integral from class

Problem: Prove that $\int_0^\infty \frac{\cos 2x}{x^4 + 1} dx = \frac{\pi\sqrt{2}}{8} e^{-\sqrt{2}} \cos \sqrt{2}$.

Solution: Since the integrand is even, we have

$$\int_0^\infty \frac{\cos 2x}{x^4 + 1} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{\cos 2x}{x^4 + 1} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{\operatorname{Re}(e^{i2x})}{x^4 + 1} dx = \frac{1}{2} \operatorname{Re} \left[\int_{-\infty}^\infty \frac{e^{i2x}}{x^4 + 1} dx \right], \quad (1)$$

so we'll compute the last integral.

The singular points of $\frac{1}{x^4 + 1}$ is the set $\{e^{i(\pi/4 + 2\pi i/4k)} : k = 0, 1, 2, 3\}$, so the only singular points above the x -axis are $z_0 = e^{i\pi/4}$, $z_1 = e^{3i\pi/4}$. In the usual notation, we have

$$\int_{-R}^R \frac{e^{i2x}}{x^4 + 1} dx + \int_{C_R} \frac{e^{i2z}}{z^4 + 1} dz = 2\pi i \left[\operatorname{Res}_{z=z_0} \frac{e^{i2z}}{z^4 + 1} + \operatorname{Res}_{z=z_1} \frac{e^{i2z}}{z^4 + 1} \right]. \quad (2)$$

We compute the residues by §76, Theorem 2, with $p(z) = e^{i2z}$, $q(z) = z^4 + 1$. We have

$$\operatorname{Res}_{z=z_0} \frac{e^{i2z}}{z^4 + 1} = \frac{e^{i2z}}{4z^3} \Big|_{z=z_0} = \frac{e^{i2e^{i\pi/4}}}{4e^{3i\pi/4}},$$

(provided $q'(z_0) \neq 0$, which is certainly the case). Thus

$$\operatorname{Res}_{z=z_0} \frac{e^{i2z}}{z^4 + 1} = \frac{e^{2i(\sqrt{2}/2 + i\sqrt{2}/2)}}{4e^{3i\pi/4}} = \frac{e^{-\sqrt{2}} e^{\sqrt{2}i}}{4e^{3i\pi/4}} = \frac{e^{-\sqrt{2}}}{4} e^{i(\sqrt{2} - 3\pi/4)}. \quad (3)$$

Similarly,

$$\operatorname{Res}_{z=z_1} \frac{e^{i2z}}{z^4 + 1} = \frac{e^{-\sqrt{2}}}{4} e^{i(-\sqrt{2} - \pi/4)}. \quad (4)$$

By (3) and (4), the RHS of (2) equals

$$\begin{aligned} \operatorname{Re} \left[2\pi i \left(\operatorname{Res}_{z=z_0} \frac{e^{i2z}}{z^4 + 1} + \operatorname{Res}_{z=z_1} \frac{e^{i2z}}{z^4 + 1} \right) \right] &= \operatorname{Re} \left[2\pi i \frac{e^{-\sqrt{2}}}{4} \left(\cos(\sqrt{2} - 3\pi/4) + i \sin(\sqrt{2} - 3\pi/4) \right. \right. \\ &\quad \left. \left. + \cos(-\sqrt{2} - \pi/4) + i \sin(-\sqrt{2} - \pi/4) \right) \right] \\ &= -2\pi \frac{e^{-\sqrt{2}}}{4} \left[(\sin \sqrt{2})(-\sqrt{2}/2) + (\cos \sqrt{2})(-\sqrt{2}/2) \right. \\ &\quad \left. + (\sin(-\sqrt{2}))(-\sqrt{2}/2) + (\cos(-\sqrt{2}))(-\sqrt{2}/2) \right] \\ &= \frac{\pi\sqrt{2}}{4} e^{-\sqrt{2}} \cos \sqrt{2}. \end{aligned} \quad (5)$$

Here we used the trig addition formula $\sin(\theta + \psi) = \cos \theta \sin \psi + \sin \theta \cos \psi$. Combining (1), (2), (5), we have

$$\int_0^\infty \frac{\cos 2x}{x^4 + 1} dx = \frac{1}{2} \operatorname{Re} \left[\int_{-\infty}^\infty \frac{e^{i2x}}{x^4 + 1} dx \right] = \frac{\pi\sqrt{2}}{8} e^{-\sqrt{2}} \cos \sqrt{2}, \quad (6)$$

provided

$$\lim_{R \rightarrow \infty} \int_{C_R} \frac{e^{i2z}}{z^4 + 1} dz = 0. \quad (7)$$

We've done this type of estimate in class. We have

$$\begin{aligned} \left| \int_{C_R} \frac{e^{i2z}}{z^4 + 1} dz \right| &= \left| \int_0^\pi \frac{e^{2iRe^{i\theta}}}{R^4 e^{i4\theta} + 1} \cdot Rie^{i\theta} d\theta \right| \\ &\leq \int_0^\pi \left| \frac{e^{2iRe^{i\theta}}}{R^4 e^{i4\theta} + 1} \right| \cdot |Rie^{i\theta}| d\theta \\ &= \int_0^\pi \frac{|e^{2iR(\cos \theta + i \sin \theta)}|}{|R^4 e^{i4\theta} + 1|} \cdot R d\theta \\ &= \int_0^\pi \frac{|e^{2iR \cos \theta}| \cdot |e^{-2R \sin \theta}|}{|R^4 e^{i4\theta} + 1|} \cdot R d\theta. \end{aligned}$$

Since $\sin \theta \in [0, 1]$ for $\theta \in [0, \pi]$, $e^{-2R \sin \theta} \in [e^{-2R}, 1]$. Also, $|e^{2iR \cos \theta}| = 1$, so

$$\left| \int_{C_R} \frac{e^{i2z}}{z^4 + 1} dz \right| \leq \int_0^\pi \frac{R}{|R^4 e^{i4\theta} + 1|} d\theta.$$

Finally,

$$|R^4 e^{i4\theta} + 1| \geq ||R^4 e^{i4\theta}| - |1|| = R^4 - 1 > R^4 - \frac{R^4}{2} = \frac{R^4}{2},$$

for $\frac{R^4}{2} > 1$, i.e. as soon as $R > \sqrt[4]{2}$. Thus

$$0 \leq \lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{e^{i2z}}{z^4 + 1} dz \right| \leq \lim_{R \rightarrow \infty} \int_0^\pi \frac{R}{R^4/2} d\theta = \frac{2\pi}{R^3} = 0,$$

so this limit is zero.