P1. §23, #3(b).

Solution: We have \( u_x = 2x, v_y = 2y, u_y = 0 = v_x \), so the C-R equations are satisfied iff \( x = y \), i.e., if \( z = x + ix \) for some \( x \in \mathbb{R} \). In this case, \( f'(z) = u_x + iv_x = 2x \).

P2. §23, #4(c).

Solution: \( u(r, \theta) = e^{-\theta} \cos(\ln r), v(r, \theta) = e^{-\theta} \sin(\ln r) \), so \( ru_r = -re^{-\theta} \sin(\ln r)(1/r) = v_\theta \) and \( u_\theta = -e^{-\theta} \cos(\ln r) = -rv_r \). Thus by the C-R polar equations, \( f \) is differentiable. \( f'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}(-e^{-\theta} \sin(\ln r)(1/r) + ie^{-\theta} \cos(\ln r)(1/r)) \). (To get the book’s formula \( f'(z) = if(z)/z \), plug in \( i = e^{i\pi} \) in the numerator and \( z = re^{i\theta} \) in the denominator.)

P3. §23, #7.

Solution: If we are given \( u_r, u_\theta \), equation (2) on p. 69 is two linear equations for the two unknowns \( u_x, u_y \). Solving for \( u_x, u_y \) gives the formulas in #7. Similarly, if we are given \( v_r, v_\theta \), using (3) and solving for \( v_x, v_y \) gives

\[
\begin{align*}
v_x &= v_r \cos \theta - v_\theta \frac{\sin \theta}{r}, \\
v_y &= v_r \sin \theta + v_\theta \frac{\cos \theta}{r}.
\end{align*}
\]

If we plug in the C-R polar equations \( ru_r = v_\theta, u_\theta = -rv_r \), we get

\[
v_y = \frac{-u_\theta}{r} \sin \theta + ru_r \frac{\cos \theta}{r} = u_r \cos \theta - \frac{u_\theta}{r} \sin \theta = u_x,
\]

and similarly we get \( u_y = -v_x \). Thus the C-R polar equations imply the usual C-R equations.

P4. §25, #7.

Solution: On \( D \), we have \( f(z) = u(x, y) = u(x, y) + i0 \), so \( f'(z) = u_x + iv_x = u_x \). But \( u_x = v_y = 0 \), so \( f'(z) = 0 \). Thus \( f(z) \) is constant.

P5. §26, #7.

Solution: We assume (usually correctly) that \( y \) is a function of \( x \) along the level curves in question, so we can write \( y = y(x) \) on each level curve. (Strictly speaking, we should write \( y = g_1(x) \) for the first level curve and \( y = g_2(x) \) on the second level curve.) Then

\[
0 = \frac{d}{dx} c_1 = \frac{d}{dx} u(x, y) = \frac{d}{dx} u(x, y(x)) = u_x + u_y y_x
\]
as in the text, and similarly $v_x + v_y y_x = 0$. Let’s rewrite this as

$$0 = u_x + u_y g_1', 0 = v_x + v_y g_2', \text{ i.e., } g_1' = -u_x/u_y, g_2' = -v_x/v_y = u_y/u_x,$$

where we use the C-R equations in the last step. Thus the slopes of $g_1$ and $g_2$ are negative reciprocals of each other at the point of intersection $z_0 = (x_0, y_0)$ of these curves, so the tangent lines are perpendicular.\(^1\)

P6. §26, #8.

**Solution:** For $f(z) = z^2 = (x + iy)^2 = (x^2 - y^2) + 2xyi$, the level curves are $x^2 - y^2 = c_1, 2xy = c_2$, which are hyperbolas for fixed $c_1 \neq 0, c_2 \neq 0$. We check that by implicit differentiation, $2x - 2y(dy/dx) = 0 \Rightarrow dy/dx = x/y$ for the first family, and $2y + 2x(dy/dx) = 0 \Rightarrow dy/dx = -y/x$ for the second family. Thus the families are perpendicular at their intersection points as long as $0 \neq f'(z_0) = 2z$. This fails precisely at $z = 0$, so the result in P5 doesn’t apply. In fact, at $z = 0$ the two level curves are $x^2 - y^2 = 0$ (or $x = \pm y$) and $2xy = 0$. The graph of the first level curve is two lines of slope $\pm 1$, and the graph of the second level curve is the $x$- and $y$-axes. Thus these level curves do not intersect perpendicularly.

---

\(^1\)Note: If $0 = f'(z_0) = u_x + iv_x$, then the computation above is invalid, since we divide by $u_x = 0$. Conversely, if $0 \neq f'(z_0)$, then either (a) $u_x \neq 0$ and $u_y \neq 0$ (which we implicitly assumed above) and we are fine, or (b) $u_x \neq 0, u_y = 0$, in which case $g_2' = 0$ and $g_1$ has vertical tangent line at the intersection point (and so the two curves are still perpendicular), or (c) $u_x = 0, u_y \neq 0$, which is similar to (b). Thus we’re ok as long as $f'(z_0) \neq 0$. For those of you who have had 411, note that in the typical case (a) $u_x \neq 0, u_y \neq 0$, we have that $g_1' \neq 0, g_2' \neq 0$, and this is precisely the hypothesis of the Implicit Function Theorem which guarantees that it is valid to write $y = g_1(x), y = g_2(x)$ for the two level curves near the intersection point.