P1. Solve $e^z = 3 - 7i$ for $z$.

Solution: $3 - 7i = \sqrt{58} e^{i\theta}$, where $\theta = \tan^{-1}(-7/3)$. Thus

$$z = \frac{1}{2} \ln 58 + i \tan^{-1}(-7/3) + 2\pi i k, k \in \mathbb{Z}.$$ 

*P2: Solve $e^z = z$ for $z = x_0 + iy_0$ by finding one equation for $y_0$ (i.e., find a function $f$ with $f(y_0) = 0$) and then solving for $x_0$ in terms of $y_0$. You won’t be able to solve these equations exactly, but use a calculator or Matlab or some other computer algebra program to find $x_0$ and $y_0$ to three decimal places.

Solution: Set program to find $x$ equations exactly, but use a calculator or Matlab or some other computer algebra ± in until I found that the roots are is symmetric with respect to the $y$-axis – this is easy to then check directly, since $f(x) = f(-x)$. I then plotted the graph between -2 and -1, and continued focusing in until I found that the roots are ±1.337 to three decimal places by looking at the graph of plot[e$^x$[x cot(x)] sin(x) - x; {x,-5,5}]” (without the quote marks). (I relabeled $y$ as $x$ just to plot this.) The part in the braces tells me I want to plot the graph between -5 and 5. I saw that there is a root between -2 and -1. I also noted that the function is symmetric with respect to the $y$-axis – this is easy to then check directly, since $f(x) = f(-x)$. I then plotted the graph between -2 and -1, and continued focusing in until I found that the roots are ±1.337 to three decimal places by looking at the graph of plot[e$^x$[x cot(x)] sin(x) - x; {x,-1.35,-1.33}]. Thus $y = ±1.337$, which implies that $x = y\cot y = .318$. (Wolfram Alpha also works as a calculator.) Thus $z = .318 ± 1.337i$ to three decimal places.

P3. §32, #2.

Solution: We know that $\ln(z_1 z_2) = \ln z_1 + \ln z_2$ as sets, but $\text{Ln}(z)$ is the specific element of $\ln z$ with imaginary part in (−π, π]. Arg(z1), Arg(z2) ∈ (−π, π], so Arg(z1) + Arg(z2) ∈ (−2π, 2π]. Since Arg(z1z2) ∈ (−π, π] and since Arg(z1z2), Arg(z2) + Arg(z2) can only differ by multiples of 2π, we can only have Arg(z1z2) − Arg(z1) − Arg(z2) = $e \in \{0, ±2\pi\}$. For example, if $z_1 = z_2 = 1$, then $e = 0$; if $z_1 = z_2 = -1$, then $e = -2\pi i$; if $z_1 = z_2 = 3\pi/2$, then $e = 2\pi i$. Since the real parts of $\ln(z_1 z_2)$ and $\ln z_1 + \ln z_2$ are equal, we must have $\ln(z_1 z_2) = \ln z_1 + \ln z_2 + 2N\pi i$ with $N \in \{0, ±1\}$.  

MA412, Problem Set #7  
Spring 2013  
Solutions
P4. Compute \((2 - i)^{2-i}\).

Solution: For \(k \in \mathbb{Z}\) (which is the same as \(-k \in \mathbb{Z}\)),

\[
(2 - i)^{2-i} = \exp[(2 - i) \ln(2 - i)] = \exp[(2 - i)\left(\frac{1}{2} \ln 5 + i \tan^{-1}(-1/2) + 2\pi ik\right)]
\]
\[
= \exp[\ln 5 + \tan^{-1}(-1/2) + 2\pi k + i((-1/2) \ln 5 + 2 \tan^{-1}(-1/2) + 4\pi k)]
\]
\[
= e^{\ln 5 + \tan^{-1}(-1/2)} e^{i((-1/2) \ln 5 + 2 \tan^{-1}(-1/2) + 4\pi k)}
\]
\[
= 5 e^{\tan^{-1}(-1/2)} e^{i((-1/2) \ln 5 + 2 \tan^{-1}(-1/2) + 4\pi k)},
\]

since \(e^{2\pi ik} = e^{4\pi ik} = 1\).

P5. What can you say about a complex number \(z\) (\(z \neq 0\)) with \(z^z = 1\)? In other words, for \(z = re^{i\theta}\), find two equations involving \(r\) and \(\theta\) whose solutions determine all \(z\) with \(z^z = 1\). (You won’t be able to solve these equations.)

Solution: \(z^z = 1\) iff \(e^{z \ln z} = 1\) iff \(z \ln z = 2\pi ik, k \in \mathbb{Z}\). For \(z = x + iy = r \cos \theta + ir \sin \theta\), this gives

\[2\pi ik = (r \cos \theta + ir \sin \theta)(\ln r + i\theta + 2\pi im),\]

for some \(m \in \mathbb{Z}\). This gives

\[
\cos \theta \ln r - \sin \theta (\theta + 2\pi m) = 0,
\]
\[
r \cos \theta (\theta + 2\pi m) + r \sin \theta \ln r = 2\pi k.
\]

We can try to solve this numerically by picking \(m\) and solving the first equation for \(\ln r\) in terms of \(\theta\), so \(\ln r = f(\theta)\) for some function \(f\). We would plug \(f(\theta)\) into the second equation for \(\ln r\), and \(e^{f(\theta)}\) into the second equation for \(r\). This would give one equation (for each value of \(m\) and \(k\)) for \(\theta\), which we solve numerically. This then determines \(r = e^{f(\theta)}\). Quite a mess.