P1. The range of $\sin x$ and $\cos x$ for $x \in \mathbb{R}$ is $[-1, 1]$. Determine the range of $\sin z$ and $\cos z$ for $z \in \mathbb{C}$.

Solution: For $\sin z$, note that $\sin^{-1} w = -i \ln[iw + (1 - w^2)^{1/2}]$ is always defined, since $iw + (1 - w^2)^{1/2}$ is never zero. (If $iw + (1 - w^2)^{1/2} = 0$, then $0 = (iw + (1 - w^2)^{1/2})^2 = (1 - 2w^2) + i2w(1 - w^2)^{1/2}$, which implies that the imaginary part vanishes, so $w = 0$ or $w = \pm 1$, but then the real part does not vanish.) So the range of $\sin$ is the domain of $\sin^{-1}$ (where $\sin^{-1}$ is an infinitely valued function), which is all of $\mathbb{C}$.

Alternatively, to see that the range of $\sin z$ is all of $\mathbb{C}$, we can show that $\sin z = w$ always has a solution. So we have to solve

$$e^{iz} - e^{-iz} = 2i, \text{ i.e., } e^{2iz} - 2iwe^{iz} - 1 = 0.$$ 

This is a quadratic equation in $e^{iz}$, so solving for $e^{iz}$ (see the solution to P2 below) and taking $\ln$ gives

$$z = -i \ln[iw + (1 - w^2)^{1/2}],$$

as it must from the first solution.

By similar arguments, the range of $\cos z$ is also all of $\mathbb{C}$.

P2: Solve for $z$: $\cos z = -2$.

Solution: We have to solve

$$e^{iz} + e^{-iz} = -2, \text{ i.e., } e^{2iz} + 4iwe^{iz} + 1 = 0.$$ 

This is a quadratic equation in $e^{iz}$ with solution $e^{iz} = -2 \pm \sqrt{3}$. Thus $iz = \ln(-2 \pm \sqrt{3})$, or $z = -i \ln(-2 \pm \sqrt{3})$. Note that $-2 \pm \sqrt{3} < 0$, so we have to write this negative number in polar form to take the log – the angle will be $\pi + 2\pi k$. For $-2 + \sqrt{3}$, we have

$$z = -i \ln(-2 + \sqrt{3}) = -i(\ln(2 - \sqrt{3}) + i(\pi + 2\pi k)) = -(\pi + 2\pi k) + i \ln(2 - \sqrt{3})^{-1},$$

and for $-2 - \sqrt{3}$, we have

$$z = -i \ln(-2 - \sqrt{3}) = -i(\ln(2 + \sqrt{3}) + i(\pi + 2\pi k)) = -(\pi + 2\pi k) + i \ln(2 + \sqrt{3})^{-1}.$$
P3. The book shows that \( \sin^{-1} z = -i \ln(i z + \sqrt{1 - z^2}) \). Prove that if \( z = x + 0i \) is purely real with \( x \in [-1, 1] \), then this definition of \( \sin^{-1} z \) coincides with the usual definition of \( \sin^{-1} x \).

**Solution:** For \( x \in [-1, 1] \), \( \sqrt{1 - x^2} \) is real, so \( ix + \sqrt{1 - x^2} \) is already in “real + imaginary” form. To take the log, we have to write this number in polar form:

\[
ix + \sqrt{1 - x^2} = re^{i\theta}.
\]

Now \( r = \sqrt{(1 - x^2) + x^2} = 1 \), and \( \theta = \sin^{-1}(y \text{-coordinate}/r) = \sin^{-1}(x/1) = \sin^{-1}(x) \). (Draw a picture of a right triangle with x-side \( \sqrt{1 - x^2} \) and y-side \( x \) and with vertex at the origin.) Thus

\[
-i \ln(i z + \sqrt{1 - z^2}) = -i \ln e^{i\sin^{-1} x} = -i(i \sin^{-1} x + 2\pi k) = \sin^{-1}(x) + 2\pi ik.
\]

This gives the answer up to a choice of \( k \); I should have said that the branch of the infinite-valued function \( \sin^{-1} z \) given by \( k = 0 \) agrees with \( \sin^{-1} x \) for \( z = x \in [-1, 1] \).

P4. Verify the formula \( \tan^{-1} z = \frac{i}{2} \log \left( \frac{i + z}{i - z} \right) \).

**Solution:** Given \( z \), we want to solve \( \tan^{-1} z = w \) or equivalently \( z = \tan w \) for \( w \). Thus

\[
z = \frac{\sin w}{\cos w} = \frac{e^{iw} - e^{-iw}}{2i} = -i e^{iw} - e^{-iw}.
\]

This gives \( e^{iw}_z + e^{-iw}_z = -i e^{iw} + ie^{-iw} \), which after multiplication by \( e^{iw} \) simplifies to

\[
e^{2iw} = \frac{i - z}{i + z},
\]

or

\[
w = \frac{1}{2i} \ln \left( \frac{i - z}{i + z} \right) = -\frac{1}{2i} \ln \left( \frac{i + z}{i - z} \right) = \frac{i}{2} \ln \left( \frac{i + z}{i - z} \right).
\]

P5. Show that the range of \( \tan z \) is \( \mathbb{C} \setminus \{\pm i\} \). *Hint:* The formula in P4 shows that \( \pm i \) are not in the range of \( \tan \). Your proof of P4 should easily show that all \( w \in \mathbb{C} \setminus \{\pm i\} \) are in the range.

**Solution:** We want to know for which \( w \) can we solve \( \tan w = z \) for \( w \). We showed in P4 that

\[
\tan w = z \iff w = \frac{i}{2} \ln \left( \frac{i + z}{i - z} \right).
\]

The RHS makes sense as long as we can take \( \ln \), which is ok unless \( \frac{i + z}{i - z} \) equals 0 or is undefined. This happens only when \( z = -i \) or \( z = i \). Thus the range of \( \tan \) is \( \mathbb{C} \setminus \{\pm i\} \).