MA563 – HOMEWORK SET 2

(1) Prove that the map $F : \mathbb{R} \longrightarrow \mathbb{R}^2$,

$$f(t) = \left(\frac{e^t + e^{-t}}{2}, \frac{e^t - e^{-t}}{2} \right),$$

is an immersion. Prove that its image is one branch of the hyperbola $x^2 - y^2 = 1$.

(2) Prove that $g : \mathbb{R} \longrightarrow \mathbb{R}^2$, $g(t) = (\cos(2\pi t), \sin(2\pi t))$ is an immersion but not an injection.

(3) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be smooth. Prove that the graph of f, given by $F : R^n \longrightarrow \mathbb{R}^{n+m}, F(x) = (x, f(x))$, is an immersion.

(4) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $g : \mathbb{R}^k \longrightarrow \mathbb{R}^\ell$ be immersions. Prove that $f \times g = (f, g) : \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^{m+\ell}$ is an immersion. If $\ell = n$, show that $f \circ g$ is an immersion.

(5) Let $p : \mathbb{C} \longrightarrow \mathbb{C}$ be a complex polynomial $p(z) = a_n z^n + a_{n-1} p^{n-1} + \ldots + a_0$ (so $z, a_n, \ldots, a_0 \in \mathbb{C}$). Considering p as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$, prove that p is a submersion except at a finite number of points.

(6) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be $f(x, y) = x^2 y^3 + \sin(xy)$. On the curve $f(x, y) = 0$ (i.e. on the set $S = \{(x, y) : f(x, y) = 0\}$), show that the Calculus I computation of dy/dx at $(x_0, y_0) \in S$ by implicit differentiation is valid iff the hypothesis of the Implicit Function Theorem is valid at (x_0, y_0)..

(7) Given functions $H, K : \mathbb{R}^3 \longrightarrow \mathbb{R}$ (so $H = H(x, y, z), K = K(x, y, z)$) and constants $a, b \in \mathbb{R}$, and a point $(x_0, y_0, z_0) \in \mathbb{R}^3$ such that $H(x_0, y_0, z_0) = a, K(x_0, y_0, z_0) = b$, find conditions on H, K such that on the set $\{(x, y, z) : H(x, y, z) = a, K = K(x, y, z) = b\}$, we know that $z = z(x, y)$ is a function of x and y on a neighborhood of (x_0, y_0, z_0). *Hint: To apply the IFT, set* $H'(x, y, z) = H(x, y, z) - a$ *and similarly for* K.

(8) Define $f : \mathbb{R}^2 \longrightarrow R$ by $f(x, y) = (y - x^2)(y - 2x^2)$. Show that the hypothesis of the Implicit Function Theorem fails at the point $(0, 0)$. Note that it is not the case that there is no function $y = y(x)$ such that $f(x, y) = 0$ near $(0, 0)$ iff $y = y(x)$, but that there are two such functions!