Homework Set 4

(1) Let c be a convex closed curve in $\mathbb{R}^2$ (i.e., c is a simple closed curve which is the boundary of a convex set.) Let $n = n(s)$ be the unit outward pointing vector to c (so $n = c'$ if c is travelled clockwise). Set $\beta(s) = c(s) + rn(s)$ for some $r > 0$. (β is called a parallel curve to c.)
   (a) Prove that the length $\ell(\beta)$ of β satisfies $\ell(\beta) = \ell(c) + 2\pi r$.
   (b) Prove that the area $A(\beta)$ enclosed by β satisfies $A(\beta) = A(c) + r\ell(c) + \pi r^2$.
   (c) Prove that $\kappa_\beta(s) = \frac{\kappa_c(s)}{1+r}$.

(2) Let $c : [a, b] \to \mathbb{R}^3$ and $\tilde{c} : [a, b] \to \mathbb{R}^3$ be regular curves. Let e_1, e_2, e_3 and $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ be Frenet frames for $c, \tilde{c}$, respectively. $\tilde{c}$ is a *Bertrand mate* of c if the normal lines of $c(t)$ and $\tilde{c}(t)$ are the same for all $t \in [a, b]$. (The normal lines are the span of $e_2(t)$ and $\tilde{e}_2(t)$.) Thus we can write $\tilde{c}(t) = c(t) + r(t)e_2(t)$, for some $r(t) \in \mathbb{R}$. Assume that both the curvature $\kappa(t)$ and the torsion $\tau(t)$ of c are never zero.
   (a) Prove that $r(t)$ is in fact a constant. Thus the Bertrand mate $\tilde{c}$ is the $\mathbb{R}^3$ version of a parallel curve to c.

   Hint: Reparametrize c by arclength. Compute that
$$\tilde{e}_1 = \frac{d\tilde{c}}{ds} = (1 - r\kappa)e_1 + r'e_2 - r\tau e_3.$$
   Since $\tilde{e}_1 \cdot e_2 = 0$ (why?), conclude that $r' = 0$.
   (b) Prove that there exist $A, B \in \mathbb{R}$, with $A, B \neq 0$, such that $A\kappa(t) + B\tau(t) = 1$ for all $t \in [a, b]$.

   Hint: Let s be the arclength parameter for c, and let $\tilde{s}$ be the arclength parameter for $\tilde{c}$. Use $d\tilde{e}_1/ds = (d\tilde{e}_1/d\tilde{s})(d\tilde{s}/ds)$ and the product rule to compute
$$\frac{d}{ds}(e_1 \cdot \tilde{e}_1) = 0.$$
   Thus $e_1 \cdot \tilde{e}_1 = \cos(\theta)$ for some constant angle θ for all s. Now use $\tilde{c} = c + rn$ (with r constant by (a)) to compute that
$$\cos(\theta) = e_1 \cdot \tilde{e}_1 = \frac{d\tilde{c}}{ds}\frac{ds}{d\tilde{s}} \cdot e_1 = \frac{ds}{d\tilde{s}}(1 - r\kappa)$$
$$|\sin(\theta)| = |\tilde{e}_1 \times e_1| = \left|\frac{ds}{d\tilde{s}}\left(e_1 + r\frac{de_2}{ds}\right) \times e_1\right| = \left|\frac{ds}{d\tilde{s}}r\tau\right|.$$
   From these equations, we get that $(1 - r\kappa)/r\tau$ is a constant, which we fix to be
$$\frac{1 - r\kappa}{r\tau} = \frac{B}{r}.$$
   Set $A = r$ and conclude that $A\kappa + B\tau = 1$.