Curve Expanding Problem

Let $\gamma(s) : [a, b] \longrightarrow \mathbb{R}^2$ be a simple closed planar curve travelled counterclockwise. Fix $r > 0$ and set $\beta(s) = \gamma(s) - r\vec{n}(s)$, where $\vec{n}$ is the unit normal to γ. Show that

(i) $\ell(\beta) = \ell(\gamma) + 2\pi r$;

(ii) $A(\beta) = A(\gamma) + r\ell + \pi r^2$, where $A(\beta), A(\gamma)$ is the area enclosed by β, γ, respectively;

(iii) $\kappa_\beta = \frac{\kappa_\gamma}{1+r\kappa_\gamma}$.

(i) First, to compute the length, we can assume that γ is a unit speed curve, and so $\ell(\gamma) = b - a.$. Then

$$\dot{\beta}(s) = \dot{\gamma}(s) - r\dot{n}(s) = \dot{\gamma}(s) + r\kappa_s\dot{\gamma}(s) = \dot{\gamma}(s)[1 + r\kappa_s].$$

Therefore,

$$\begin{aligned}
\ell(\beta) &= \int_a^b |\dot{\beta}(s)|ds = \int_a^b |\dot{\gamma}(s)|(1 + r\kappa_s)ds \\
&= \int_a^b (1 + r\frac{d\phi}{ds}ds = (b - a) + r\int_a^b \frac{d\phi}{ds}ds \\
&= \ell(\gamma) + r \cdot 2\pi
\end{aligned}$$

by the Turning Tangent Theorem.

(ii) If $\gamma(s) = (x(s), y(s))$, then by Green's Theorem,

$$A(\gamma) = \frac{1}{2}\int_a^b -ydx + xdy = \frac{1}{2}\int_a^b -y\dot{x} + x\dot{y}ds.$$

We have $\dot{\gamma}(s) = (\dot{x}(s), \dot{y}(s))$ (with $(\dot{x})^2 + (\dot{y})^2 = 1$), and so $n(s) = (-\dot{y}(s), \dot{x}(s))$, since this last vector has length one, is perpendicular to $\dot{\gamma}$, and points in the right direction. Thus

$$\beta(s) = (x(s) + r\dot{y}(s), y(s) - r\dot{x}(s)),$$

and the area enclosed by β is

$$\begin{aligned}
A(\beta) &= \frac{1}{2}\int_a^b -(y - r\dot{x})(\dot{x} + r\ddot{y}) + (x + r\dot{y})(\dot{y} - r\ddot{x})ds \\
&= \frac{1}{2}\left[\int_a^b (x\dot{y} - y\dot{x})ds + r\int_a^b (\dot{x})^2 + (\dot{y})^2 + r^2\int_a^b (\ddot{y}\dot{x} - \dot{y}\ddot{x})ds - r\int_a^b (\ddot{y}y + \ddot{x}x)ds\right] \\
&= A(\gamma) + \frac{1}{2}r\ell(\gamma) + \frac{r^2}{2}\int_a^b (\ddot{y}\dot{x} - \dot{y}\ddot{x})ds - \frac{r}{2}\int_a^b (\ddot{y}y + \ddot{x}x)ds. \qquad (0.1)
\end{aligned}$$

Now $\dot{n} = -\kappa_s\vec{t} = -\kappa_s\dot{\gamma}$, so $\dot{n}\cdot\dot{\gamma} = -\kappa_s$, and by the formulas above, we get $\kappa_s = \dot{x}\ddot{y} - \ddot{x}\dot{y}$. Thus by the Turning Tangent Theorem, the third integral on the last line of (**??**)

becomes

$$\frac{1}{2}r^2\int_a^b(\ddot{y}\dot{x}-\dot{y}\ddot{x})ds=\frac{1}{2}r^2\int_a^b\kappa_s ds=\frac{1}{2}r^2\int_a^b\frac{d\phi}{ds}ds=\pi r^2.$$

By integration by parts, the last integral in (**??**) is

$$-\frac{r}{2}\int_a^b(\ddot{y}y+\ddot{x}x)ds=\frac{r}{2}\int_a^b(\dot{y}\dot{y}+\dot{x}\dot{x})ds+\frac{r}{2}\left[\dot{x}x+\dot{y}y\right]_a^b=\frac{r}{2}\int_a^b 1ds=\frac{r}{2}\ell(\gamma),$$

since $x(a)=x(b), y(a)=y(b)$. Thus

$$A(\beta)=A(\gamma)+r\ell(\gamma)+\pi r^2.$$

(iii) We've computed $\dot{\beta}(s)=\dot{\gamma}(s)[1+r\kappa_s(\gamma)]$. Dropping the s's and writing κ for $\kappa_s(\gamma)$, this implies

$$\ddot{\beta}=\dot{\gamma}r\dot{\kappa}+\ddot{\gamma}[1+r\kappa]=\dot{\gamma}r\dot{\kappa}+\kappa[1+r\kappa]n.$$

It follows easily that $|\ddot{\beta}\times\dot{\beta}|=\kappa[1+r\kappa]^2$, and that

$$\kappa(\beta)=\frac{\kappa(1+r\kappa)^2}{(1+r\kappa)^3}=\frac{\kappa}{1+r\kappa}.$$