Curve Expanding Problem

Let $\gamma(s) : [a, b] \rightarrow \mathbb{R}^2$ be a simple closed planar curve travelled counterclockwise. Fix $r > 0$ and set $\beta(s) = \gamma(s) - r\vec{n}(s)$, where $\vec{n}$ is the unit normal to $\gamma$. Show that

(i) $\ell(\beta) = \ell(\gamma) + 2\pi r$;
(ii) $A(\beta) = A(\gamma) + r\ell + \pi r^2$, where $A(\beta), A(\gamma)$ is the area enclosed by $\beta, \gamma$, respectively;
(iii) $\kappa_\beta = \frac{\kappa_\gamma}{1 + r\kappa_\gamma}$.

(i) First, to compute the length, we can assume that $\gamma$ is a unit speed curve, and so $\ell(\gamma) = b - a$. Then

$$\dot{\beta}(s) = \dot{\gamma}(s) - r\dot{n}(s) = \dot{\gamma}(s) + r\kappa_\gamma\dot{\gamma}(s) = \dot{\gamma}(s)[1 + r\kappa_\gamma].$$

Therefore,

$$\ell(\beta) = \int_a^b |\dot{\beta}(s)| ds = \int_a^b |\dot{\gamma}(s)| (1 + r\kappa_\gamma) ds$$

$$= \int_a^b (1 + r\frac{d\phi}{ds}) ds = (b - a) + r \int_a^b \frac{d\phi}{ds} ds$$

$$= \ell(\gamma) + r \cdot 2\pi$$

by the Turning Tangent Theorem.

(ii) If $\gamma(s) = (x(s), y(s))$, then by Green’s Theorem,

$$A(\gamma) = \frac{1}{2} \int_a^b -ydx + xdy = \frac{1}{2} \int_a^b -y\dot{x} + x\dot{y} ds.$$

We have $\dot{\gamma}(s) = (\dot{x}(s), \dot{y}(s))$ (with $(\dot{x})^2 + (\dot{y})^2 = 1$), and so $n(s) = (-\dot{y}(s), \dot{x}(s))$, since this last vector has length one, is perpendicular to $\dot{\gamma}$, and points in the right direction. Thus

$$\beta(s) = (x(s) + r\dot{y}(s), y(s) - r\dot{x}(s)),$$

and the area enclosed by $\beta$ is

$$A(\beta) = \frac{1}{2} \int_a^b -(y - r\dot{x})(\dot{x} + r\dot{y}) + (x + r\dot{y})(\dot{y} - r\dot{x}) ds$$

$$= \frac{1}{2} \left[ \int_a^b (x\dot{y} - y\dot{x}) ds + r \int_a^b (\dot{y})^2 + r^2 \int_a^b (\ddot{y} - \ddot{x}) ds - r \int_a^b (\dot{y}\ddot{x} - \dot{x}\ddot{y}) ds \right]$$

$$= A(\gamma) + \frac{1}{2} r\ell(\gamma) + \frac{r^2}{2} \int_a^b (\ddot{y} - \ddot{x}) ds - \frac{r}{2} \int_a^b (\dot{y}\ddot{x} - \dot{x}\ddot{y}) ds. \quad (0.1)$$

Now $\dot{n} = -\kappa_\gamma\vec{t} = -\kappa_\gamma\dot{\gamma}$, so $\dot{n} \cdot \dot{\gamma} = -\kappa_\gamma$, and by the formulas above, we get $\kappa_\gamma = \dot{x}\ddot{y} - \dot{y}\ddot{x}$. Thus by the Turning Tangent Theorem, the third integral on the last line of (??)
becomes
\[
\frac{1}{2} r^2 \int_a^b (\dot{y} \ddot{x} - \dot{x} \ddot{y}) ds = \frac{1}{2} r^2 \int_a^b \kappa_s ds = \frac{1}{2} r^2 \int_a^b \frac{d\phi}{ds} ds = \pi r^2.
\]
By integration by parts, the last integral in (??) is
\[
-\frac{r}{2} \int_a^b (\dot{y} \ddot{y} + \ddot{x} \dot{x}) ds = \frac{r}{2} \int_a^b (\dot{y} \ddot{y} + \ddot{x} \dot{x}) ds + \frac{r}{2} [\dddot{x} + \dddot{y}]_a^b = \frac{r}{2} \int_a^b 1 ds = \frac{r}{2} \ell(\gamma),
\]
since \(x(a) = x(b), y(a) = y(b)\). Thus
\[
A(\beta) = A(\gamma) + r\ell(\gamma) + \pi r^2.
\]
(iii) We’ve computed \(\dot{\beta}(s) = \dot{\gamma}(s)[1 + r\kappa_s(\gamma)]\). Dropping the \(s\)’s and writing \(\kappa\) for \(\kappa_s(\gamma)\), this implies
\[
\ddot{\beta} = \dot{\gamma} \kappa + \ddot{\gamma} [1 + r\kappa] = \dot{\gamma} \kappa + \kappa [1 + r\kappa] n.
\]
It follows easily that \(|\ddot{\beta} \times \dot{\beta}| = \kappa [1 + r\kappa]^2\), and that
\[
\kappa(\beta) = \frac{\kappa(1 + r\kappa)^2}{(1 + r\kappa)^3} = \frac{\kappa}{1 + r\kappa}.
\]