ORIENTATION PRESERVING ISOMETRIES OF $\mathbb{R}^2$

Let $\text{Isom}_o(\mathbb{R}^2)$ be the set of orientation preserving isometries of $\mathbb{R}$. Thus $f \in \text{Isom}_o(\mathbb{R}^2)$ implies $d(\vec{x}, \vec{y}) = d(f\vec{x}, f\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^2$.

(1) Show that $f \in \text{Isom}_o(\mathbb{R}^2)$ is continuous.

If $\vec{x}_i \to \vec{x}$ in $\mathbb{R}^2$, then

$$d(\vec{x}_i, \vec{x}) \to 0 \Rightarrow d(f\vec{x}_i, f\vec{x}) \to 0 \Rightarrow f\vec{x}_i \to f\vec{x}.$$

This is one formulation of the continuity of f.

(2) For $f \in \text{Isom}_o(\mathbb{R}^2)$, show that $f = T_{\vec{v}} \circ R$, where $T_{\vec{v}}$ is translation by $\vec{v} = f(\vec{0})$, and R is an orientation preserving isometry with $R(\vec{0}) = \vec{0}$.

$T_{\vec{v}}$ is easily an isometry, as $d(T_{\vec{v}}(\vec{x}), T_{\vec{v}}(\vec{y})) = d(\vec{x}+\vec{v}, \vec{y}+\vec{v}) = d(\vec{x}, \vec{y})$ by the usual distance formula in $\mathbb{R}$. Since f is an isometry and the inverse of $T_{\vec{v}}$ is $T_{-\vec{v}}$, which is also an isometry, we get $R = T_{-\vec{v}} \circ f$ is an isometry. Also $R(\vec{0}) = (T_{-\vec{v}} \circ f)(\vec{0}) = T_{-\vec{v}}(\vec{v}) = \vec{0}$. Finally, since translations are orientation preserving and f is orientation preserving, the composition $R = T_{-\vec{v}} \circ f$ is orientation preserving.

(3) Show that R is a linear transformation.

The triangle inequality states that $d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z}) \geq d(\vec{x}, \vec{z})$ with equality iff $\vec{x}, \vec{y}, \vec{z}$ are collinear with $\vec{y}$ between $\vec{x}$ and $\vec{z}$. Thus if $\vec{x}, \vec{y}, \vec{z}$ are collinear, so are $R\vec{x}, R\vec{y}, R\vec{z}$, since $d(\vec{x}, \vec{y}) = d(R\vec{x}, R\vec{y})$, etc. implies we still have equality in the triangle inequality for $R\vec{x}, R\vec{y}, R\vec{z}$. This easily implies that R takes lines to lines.

Since $\vec{0}, \vec{x}, \lambda\vec{x}$ are collinear, so are $\vec{0} = R\vec{0}, R\vec{x}, R(\lambda\vec{x})$. But $\vec{0}, R\vec{x}, \lambda R\vec{x}$ are also collinear. Looking at the distances between $\vec{0}, R\vec{x}, \lambda R\vec{x}, R(\lambda\vec{x})$, you can easily check that we must have

$$R(\lambda\vec{x}) = \lambda R(\vec{x}).$$

Now let $\vec{x}, \vec{y}$ be linearly independent. By the usual picture of vector addition, $\vec{x} + \vec{y}$ is the diagonal of the parallelogram P_1 determined by $\vec{x}, \vec{y}$, and $R\vec{x} + R\vec{y}$ is the diagonal of the parallelogram P_2 determined by $R\vec{x}, R\vec{y}$. Since R is an isometry fixing $\vec{0}$, it must take P_1 to P_2, and so it takes diagonal to diagonal:

$$R(\vec{x} + \vec{y}) = R\vec{x} + R\vec{y}.$$

Thus R is a linear transformation.

(4) Show that R is a rotation by some angle.

By looking at congruent triangles, isometries must preserve angles. Since $\vec{v} \cdot \vec{w} = |\vec{v}|\ |\vec{w}| \cos(\theta)$, isometries preserve the dot product: $\vec{v} \cdot \vec{w} = R\vec{v} \cdot R\vec{w}$. Thus $R\vec{i}$ is perpendicular to $R\vec{j}$. Since R is orientation preserving, $R\vec{j}$ must be a 90 degree counterclockwise rotation of $R\vec{i}$. Thus if $R\vec{i}$ is rotation of $\vec{i}$ by angle θ, $R\vec{j}$ is rotation of $\vec{j}$ by θ. Since R is a linear transformation, by the parallelogram picture it easily follows that R rotates all vectors by θ.

(5) Lef F_x be the flip on the x-axis: $F_x(x, y) = (x, -y)$. Show that every isometry has a unique decomposition of the form $f = TRF$, where T is a translation, R is a rotation around the origin, and F is either the identity or F_x.

If f is orientation preserving, we have already written $f = TR$. If f is orientation reversing, the $f \circ F_x$ is orientation preserving, so $f \circ F_x = TR$. Thus $f = f \circ f_x \circ F_x = TRF_x$.

If $T_1R_1F_1 = T_2R_2F_2$, then $R_2^{-1}T_2^{-1}T_1R_1 = F_2F_1$. The left hand side is orientation preserving, so F_2F_1 is orientation preserving. Thus F_1, F_2 are either both the identity or both F_x.

Thus we get $T_1R_1F_1 = T_2R_2F_2 = T_2R_2F_1$, so $T_1R_1 = T_2R_2$. This implies $T_2^{-1}T_1 = R_2R_1^{-1}$. The right hand side takes $\vec{0}$ to $\vec{0}$. The left hand side is a translation, so it must be a translation by zero. Thus $T_1 = T_2$.

So now $T_1R_1 = T_2R_2 = T_1R_2$. This implies $R_1 = R_2$.