

Geometry and Symmetry, Problem Set #1
Summer 2009

Exploration

P1: Label each edge and each vertex of an equilateral triangle with one of the numbers 1 through 6 so that each number is used exactly once. We call the result a “magic triangle” if the sum is the same along each of the three edges (i.e. if the sum of the numbers attached to an edge and its two endpoints is the same for all three edges). How many different magic triangles are there? Please explain carefully what you mean by “different”.

Lines in the plane

A *line* in the plane \mathbb{R}^2 is by definition the set $\{(x, y) \in \mathbb{R}^2 \mid ax + by = c\}$ for some fixed choice of $a, b, c \in \mathbb{R}$ (with at least one of a, b nonzero). The expression $ax + by = c$ is the *equation* of the line.

P2. What is the equation of the line ℓ_1 in \mathbb{R}^2 joining $(0, 0)$ and $(1, 2)$? Is this equation unique? The line ℓ_2 joining $(-4, 3)$ and $(1, 2)$? The line ℓ_3 joining $(-4, 3)$ and $(-4, 2)$? What is $\ell_1 \cap \ell_2, \ell_1 \cap \ell_3, \ell_1 \cap \ell_2 \cap \ell_3$?

P3. PODASIP (Prove or Disprove and Salvage if Possible): Every line in \mathbb{R}^2 can be written as $y = mx + b$ for a unique $m, b \in \mathbb{R}$.

P4. Let $\vec{v} = (1, 2), \vec{w} = (-2, 5)$. What is $2\vec{v}, \vec{v} - \vec{w}, 4\vec{v} - 5\vec{w}$? What (geometric shape) is (described by) $\{\lambda\vec{w} : \lambda \in \mathbb{R}\}$? What is $\{\vec{v} + \lambda\vec{w} : \lambda \in \mathbb{R}\}$? Find vectors \vec{v}, \vec{w} such that the line ℓ_2 of P2 is given by $\{\vec{v} + \lambda\vec{w} : \lambda \in \mathbb{R}\}$. Are \vec{v}, \vec{w} unique? Do the same for ℓ_1, ℓ_3 .

P5. PODASIP: Every line in \mathbb{R}^2 can be written as $\{\vec{v} + \lambda\vec{w} : \lambda \in \mathbb{R}\}$. *Hint:* What if $\vec{w} = \vec{0}$?

P6. PODASIP: Two lines $\{(x, y) \mid a_0x + b_0y = c_0\}, \{(x, y) \mid a_1x + b_1y = c_1\}$ are parallel iff $a_0 = a_1$ and $b_0 = b_1$.

Lines and planes in space

P7. Let $\vec{v} = (1, 2, -2), \vec{w} = (-2, 5, 4)$. What is $2\vec{v}, \vec{v} - \vec{w}, 4\vec{v} - 5\vec{w}$? What (geometric shape) is (described by) $\{\lambda\vec{w} : \lambda \in \mathbb{R}\}$? What is $\{\vec{v} + \lambda\vec{w} : \lambda \in \mathbb{R}\}$? What is $\{\lambda\vec{v} + \mu\vec{w} : \lambda, \mu \in \mathbb{R}\}$? What is the set of vectors of the form $(2, 3, 7) + \lambda\vec{v} + \mu\vec{w}$, for $\lambda, \mu \in \mathbb{R}$?

P8. Find vectors \vec{v}, \vec{w} such that the line through $(-1, 3, 0)$ and $(3, -3, 4)$ is given by $\{\vec{v} + \lambda\vec{w} : \lambda \in \mathbb{R}\}$. Are \vec{v}, \vec{w} unique? Do the same for the points $(0, 0, 0)$ and $(1, 4, 6)$.

P9. Which definition of a line in \mathbb{R}^2 extends to give a good definition of a line in \mathbb{R}^3 : the set $\{(x, y) \mid ax + by = c\}$ or the set $\{\lambda\vec{v} + \vec{w} : \lambda \in \mathbb{R}\}$? Why?

P10. Find a point (a, b, c) and vectors \vec{v}, \vec{w} such that the plane containing $(1, 2, -1), (2, 3, 4), (-3, 4, 1)$ is given by the set of vectors $(a, b, c) + \lambda\vec{v} + \mu\vec{w}$, for $\lambda, \mu \in \mathbb{R}$.

P11. Find $a, b, c, d \in \mathbb{R}$ such that the three points $(1, 2, -1), (2, 3, 4), (-3, 4, 1)$ of P10 satisfy $ax + by + cz = d$. Are a, b, c, d unique?

P12. Show that every set in \mathbb{R}^3 of the form $\{(x, y, z) : ax + by + cz = d\}$, for some choice of a, b, c, d , can be written in the form $\{\vec{v} + \lambda\vec{w} + \mu\vec{u} \mid \lambda, \mu \in \mathbb{R}\}$, for some choice of $\vec{v}, \vec{w}, \vec{u}$, and *vice versa*.

You can use either set $\{(x, y, z) : ax + by + cz = d\}, \{\vec{v} + \lambda\vec{w} + \mu\vec{u} \mid \lambda, \mu \in \mathbb{R}\}$ as the definition of a *plane* in \mathbb{R}^3 . The expression $ax + by + cz = d$ is the *equation* of the plane.

- P13. What is the equation of the plane P_1 in \mathbb{R}^3 containing $(0, 0, 0)$, $(1, 2, 3)$ and $(2, 3, 4)$? The plane P_2 containing $(0, 0, 0)$, $(1, 2, 3)$ and $(2, 4, 6)$? The plane P_3 containing $(-1, 0, 0)$, $(1, 2, 3)$ and $(2, 4, 6)$? Write $P_1 \cap P_3$ as $\{\vec{v} + \lambda\vec{w} \mid \lambda \in \mathbb{R}\}$ for vectors \vec{v}, \vec{w} .
- P14. Find two planes whose intersection is the line joining $(0, 2, 2)$ and $(1, 2, 3)$. Are these planes unique?
- P15. Show that two distinct intersecting planes in \mathbb{R}^3 intersect in a line, using either of our definitions of a plane.
- P16. Find the equation of the plane parallel to $x - 2y + 3z = 7$ passing through the origin.

Synthetic vs. Analytic Geometry

- P17. Prove that the diagonals of a parallelogram bisect each other synthetically – i.e. you may use the usual congruence theorems (SAS, AAS, etc.) and other theorems of high school geometry, but no coordinates allowed.
- P18. Prove that the diagonals of a parallelogram bisect each other analytically – i.e. using coordinates in the plane. To make the calculations simple, justify that you can put one vertex at the origin and one side along the x -axis. Which proof is easier?
- P19. Prove the triangle inequality analytically: for points $A, B, C \in \mathbb{R}^2$, $d(A, B) + d(B, C) \geq d(A, C)$. Is there a synthetic proof?

A Challenge Problem

- P20. Let P, Q be two points inside the unit circle S^1 which do not lie on a diameter of S^1 . Show that there exists a unique circle containing P and Q which hits S^1 at right angles. You can give either an analytic or synthetic proof.