

Geometry and Symmetry, Problem Set 10
Summer 2009

Exploration – Symmetries of Platonic Solids

- P1. How many symmetries are there of the regular tetrahedron? (A symmetry is an isometry of \mathbb{R}^3 that takes vertices of the tetrahedron to vertices.) How many preserve orientation? How many reverse orientation? What is the stabilizer subgroup of a fixed vertex? Consider each symmetry as a bijection of the set of six edges of the tetrahedron. What is the stabilizer of a fixed edge?
- P2. Answer the same questions for the cube.

Paradoxical Groups

- P3. Show that a finite group is not very paradoxical.
- P4. Given vectors $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$ not lying on a line through the origin, and another pair of vectors $\vec{w}_1, \vec{w}_2 \in \mathbb{R}^2$ not lying on a line through the origin, show that there exists a unique $C \in \text{Gl}(2, \mathbb{R})$ with $C\vec{v}_1 = \vec{w}_1, C\vec{v}_2 = \vec{w}_2$. *Hint:* You can solve four equations in four unknowns if you must. More geometrically, let A be the matrix taking \vec{i} to v_1 and \vec{j} to v_2 , and let B be the matrix taking \vec{i} to w_1 and \vec{j} to w_2 . Find C in terms of A and B , which are easy to write down.
- P5. Let \vec{v}_1, \vec{v}_2, A be as in P4. Show that the interior of the infinite wedge shaped region bounded by the rays $\{\lambda\vec{v}_1 : \lambda \geq 0\}, \{\lambda\vec{v}_2 : \lambda \geq 0\}$ is taken by A to the interior of the region bounded by the rays $\{\lambda\vec{w}_1 : \lambda \geq 0\}, \{\lambda\vec{w}_2 : \lambda \geq 0\}$. (Here the interior means the region cut out by the angle between the rays which is less than π .) *Hint:* Show that a vector \vec{v} is in the first region iff $\vec{v} = \lambda\vec{v}_1 + \mu\vec{v}_2$ with $\lambda \geq 0, \mu \geq 0$.
- P6. Show that you can find three disjoint infinite wedges A_1, A_2, A_3 as in P5 and $C_1, C_2, C_3 \in \text{Gl}(2, \mathbb{R})$ such that $\mathbb{R}^2 = C_1(A_1) \cup C_2(A_2) \cup C_3(A_3)$. (*Hint:* Say the boundary of A_1 are rays determined by \vec{v}_1, \vec{v}_2 . Pick \vec{w}_1, \vec{w}_2 whose wedge is almost all of the upper half plane. Find C_1 as in P4. Repeat this for A_2 so that $C_2(A_2)$ covers almost all of the lower half plane. Find C_3, A_3 so that $C_3(A_3)$ covers the rest of \mathbb{R}^2 . Be careful to include the origin in only one of your wedges.) Conclude that \mathbb{R}^2 is $\text{Gl}(2, \mathbb{R})$ -paradoxical.
- P7. Is F_3 paradoxical? Is \mathbb{Z} ? *Hint:* for \mathbb{Z} , show that one of the “A” sets contains some interval $[N, \infty)$ for some $N \in \mathbb{Z}$, and that one of the “B” sets contains a similar interval $[N', \infty)$.

Group Actions

- P8. Show that $\text{SO}(3)$ acts on \mathbb{R}^3 . What is the orbit of $\vec{v} = (1, 0, 0)$? Of $\vec{w} = (0, 1, 0)$? Of $\vec{0} = (0, 0, 0)$? What is the stabilizer $S_{\vec{v}}$? What is $S_{\vec{w}}$? What is $S_{\vec{0}}$? If $A\vec{v} = B\vec{v}$, does it follow that $A = B$? What can you say about such A and B ?
- P9. Redo P8 with $\text{SO}(3)$ replaced by $\text{GL}(3, \mathbb{R})$.
- P10. Show that the unit quaternions S^3 act on \mathbb{R}^3 (thought of as the imaginary quaternions I) by conjugation: $q \cdot v = C_q(v) = qvq^{-1}$. Find the orbits for this action.
- P11. Show that $SL(2, \mathbb{R})$ acts on \mathbb{H} via linear fractional transformations: if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$, then $A \cdot z = \frac{az+b}{cz+d}$. Show that this action is transitive.

- P12. For the action in P11, what is the stabilizer subgroup of i ? What is the stabilizer subgroup of $1+2i$? Find $A \in SL(2, \mathbb{R})$ such that $A \cdot i = 1+2i$. Show that $B \cdot i = i$ iff $ABA^{-1} \cdot (1+2i) = 1+2i$.
- P13. PODASIP: All stabilizer subgroups S_z of points $z \in \mathbb{H}$ are isomorphic. All stabilizer subgroups S_x of points $x \in X$ are isomorphic for an arbitrary group action of a group G on a set X .
- P14. Define an action of $GL(3, \mathbb{R})$ on \mathbb{RP}^2 as follows: if $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ w \\ q \end{pmatrix}$, set $A \cdot [x, y, z] = [z, w, q]$. Show that this map is well defined and gives a group action of $GL(3, \mathbb{R})$ on \mathbb{RP}^2 . What is the orbit and stabilizer of each point?
- P15. As in class, add a line at infinity to \mathbb{Z}_3^2 to get $\mathbb{Z}_3\mathbb{P}^2$. Define an action of $GL(3, \mathbb{Z}_3)$ on $\mathbb{Z}_3\mathbb{P}^2$ as follows: if $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ w \\ q \end{pmatrix}$, set $A \cdot [x, y, z] = [z, w, q]$. Show that this map is well defined and gives a group action of $GL(3, \mathbb{Z}_3)$ on $\mathbb{Z}_3\mathbb{P}^2$. What is the orbit and stabilizer of each point?
- P16. Find slices for the group actions in P8–11, 15, 16.
- *P17. Write $S^1 = \{a + bi : a, b \in \mathbb{R}, a^2 + b^2 = 1\}$, thought of as a subset of the unit quaternions $S^3 = \{a + bi + cj + dk : a^2 + b^2 + c^2 + d^2 = 1\}$. We can also think of \mathbb{R}^4 as \mathbb{C}^2 , so $S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$. Show that S^3 acts transitively on the unit sphere S^2 inside the imaginary quaternions by conjugation: $q \cdot v = qvq^{-1}$. Show that the stabilizer of i is $\{a + bi : a^2 + b^2 = 1\} = S^1$, so we get a bijection $\alpha : S^3/S^1 \rightarrow S^2 = \mathbb{CP}^1$. Show that the map $S^3 \rightarrow S^2$ given by $q \mapsto \alpha[q]$ is the “Hopf map” $h : S^3 \rightarrow S^2$, $h(z, w) = [z, w]$.