

Geometry and Symmetry, Problem Set 11
Summer 2009

Exploration – The Upper Half Plane and the Unit Disk

- P1. Let $B = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ be the complex linear fractional transformation $B \cdot z = \frac{z-i}{z+i}$. Show that if $z \in \mathbb{H}$, then $|B \cdot z| < 1$, so $B : \mathbb{H} \rightarrow \mathcal{D}$, the unit disk. What is $B \cdot 0$? What is $B \cdot i\infty$, i.e. $\lim_{y \rightarrow \infty} B \cdot iy$? Show that B is a bijection by finding an inverse for B . (*Hint*: Solve $w = \frac{z-i}{z+i}$ for z .) Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Show that $BSB^{-1} : \mathcal{D} \rightarrow \mathcal{D}$ is a rotation by π around the origin.

Numericals with $SL(2, \mathbb{Z})$

- P2. Set $SL(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) : a, b, c, d \in \mathbb{Z} \right\}$. Show that $SL(2, \mathbb{Z})$ is a subgroup of $SL(2, \mathbb{R})$. Show that the action of $SL(2, \mathbb{R})$ on \mathbb{H} is transitive, but the action of $SL(2, \mathbb{Z})$ is not transitive.
- P3. Let $\mathcal{F} \subset \mathbb{H}$ be defined by $\mathcal{F} = \{z \in \mathbb{C} : |z| \geq 1, |\operatorname{Re}(z)| \leq \frac{1}{2}\}$. Draw a picture of \mathcal{F} . Define $S, T \in SL(2, \mathbb{Z})$ by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Which linear fractional transformations correspond to S and T ? What is the order of S as a matrix? What is the order of S as a linear fractional transformation? What does $S(\mathcal{F})$ look like? What does $T(\mathcal{F})$ look like? What about $S^{-1}(\mathcal{F}), T^{-1}(\mathcal{F}), ST(\mathcal{F}), TS(\mathcal{F}), S^{17}T^{-3}ST^{44}(\mathcal{F})$?
- P4. Let G be the group generated by S and T , so elements of G are given by strings in $S^{\pm 1}, T^{\pm 1}$. Find some obvious relations (swear words) in G . (*Hint*: What is S^2 ? Is $ST = TS$?) For $z = 4 + \frac{i}{2}$, find $A \in SL(2, \mathbb{Z})$ such that $A \cdot z$ is in \mathcal{F} . Do the same for $z = \frac{7}{8} + \frac{4}{5}i$. Can you write A as an element of G ?
- P5. Show that every element of G is in $SL(2, \mathbb{Z})$. Write $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ as an element of G – i.e. as a word in $S^{\pm 1}, T^{\pm 1}$. (*Hint*: Pick $z \in \mathcal{F}$ and figure out what combination of S and T takes $A \cdot z$ back into \mathcal{F} .) Do the same for $B = \begin{pmatrix} 9 & 7 \\ 5 & 4 \end{pmatrix}$. Any interesting number theory involved?
- P6. Find a word W in $T^{\pm 1}, S^{\pm 1}$ such that $W = \begin{pmatrix} 4 & 5 \\ * & * \end{pmatrix}$, i.e. with top row 4 5. Do the same for $\begin{pmatrix} 9 & 7 \\ * & * \end{pmatrix}$ and $\begin{pmatrix} 37 & -19 \\ * & * \end{pmatrix}$. Since these matrices all have determinant one, you've solved equations like $37d + 19c = 1$ for integers d, c . Can you systemize your search for W ? Is your algorithm different from Euclid's algorithm, or is it essentially the same?

Group Actions

- P7. Let a finite group G act transitively on a set X , and let S_x be the stabilizer subgroup of some $x \in X$. Check that $|X| < \infty$. Show that $|G| = |X| \cdot |S_x|$.
- P8. Check your answers to Set 10, P1 and P2 using the previous problem.
- P9. Compute the order of the symmetry groups of the octahedron, icosahedron and dodecahedron.

Paradoxical Decompositions

- P10. Let G act via isometries on a set X with a distance function. Let $A, B \subset X$. Prove that the relation $A \sim_G B$ iff A and B are G -equidecomposable is an equivalence relation on subsets of X .
- P11. PODASIP: Let A, B be rectangles in \mathbb{R}^2 with the same area. We can chop up A into a finite number of rectangles and move the rectangles by isometries to exactly fill out B .
- P12. Let $O(n+1)$ be the group of isometries of \mathbb{R}^{n+1} fixing the origin. Show that S^n is $O(n)$ -paradoxical for $n \geq 2$. *Hint:* We will do the case $n = 2$ in class using the existence of an injective homomorphism $\phi : F_2 \rightarrow \text{SO}(3) \subset O(3)$. Find an injective homomorphism $\phi' : \text{SO}(3) \rightarrow O(n+1)$. (Think of $\text{SO}(2)$ sitting inside of $O(3)$ as the rotations of the xy -plane.) This gives a copy of F_2 inside $O(n+1)$, and we can repeat the proof from class. (The proof works for $\text{SO}(n+1)$, but it's a pain to describe orientation preserving/reversing transformations in \mathbb{R}^{n+1} .)
- P13. Show that $\mathbb{R}^n \setminus \{0\}$ is $\text{SO}(n)$ -paradoxical for $n \geq 3$. *Hint:* By P12, there are sets $A_1, \dots, A_n, B_1, \dots, B_m$ giving a paradoxical decomposition of S^{n-1} . Let A'_i be all the rays from the origin through points of A_i excluding the origin, and define B'_j similarly. Show that the A'_i, B'_j give a paradoxical decomposition of $\mathbb{R}^n \setminus \{0\}$.
- P14. Show that \mathbb{R}^n is $\text{Isom}(\mathbb{R}^n)$ -paradoxical for $n \geq 3$. *Hint:* We want to mimic the proof that $S^2 \setminus D \sim_{\text{SO}(3)} S^2$ for a countable set D to show that $\mathbb{R}^n \setminus \{0\} \sim_{\text{Isom}(\mathbb{R}^n)} \mathbb{R}^n$. Consider rotations of the one point set $\{0\}$ around a line not through the origin through powers of a “bad” angle..

Linear Fractional Transformations and \mathbb{CP}^1

- P15. We can define a “linear transformation” on \mathbb{CP}^1 by $A([z, w]) = [az + bw, cz + dw]$ for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with all entries complex numbers. (The coordinates of $A([z, w])$ are linear functions of z, w , hence the name.) Consider \mathbb{H} as a subset of \mathbb{C} , the set of finite points of \mathbb{CP}^1 . Then for $z \in \mathbb{H}$, A determines two actions on z , the linear fractional transformation $A \cdot z = \frac{az+b}{cz+d}$ and $A([z, 1])$. Show that these two transformations are the same. So linear fractional transformations come from linear transformations of \mathbb{CP}^1 .
- P16. For $A \in \text{SL}(2, \mathbb{R})$, set $A \cdot i\infty = \lim_{y \rightarrow \infty} A \cdot iy$. Since $i\infty$ also corresponds to $[1, 0] \in \mathbb{CP}^1$, we can also set $A \cdot i\infty = A([1, 0])$. Show that these two definitions agree. Define $S, T \in \text{SL}(2, \mathbb{Z})$ as in P3. What is $S(i\infty)$? What is $T(i\infty), ST(i\infty), TS(i\infty), T^{-7}ST^2S(i\infty)$? What are the possible values of $W(i\infty)$ for $W \in G$, the group in P4? Can you find a word W such that $W(i\infty) = 943/19$? What is $\lim_{n \rightarrow \infty} (TS)^n(i\infty)$?