

**Geometry and Symmetry, Problem Set 11**  
**Summer 2011**

**Exploration – The Upper Half Plane and the Unit Disk**

- P1. Let  $B = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$  be the complex linear fractional transformation  $B \cdot z = \frac{z-i}{z+i}$ . Show that if  $z \in \mathbb{H}$ , then  $|B \cdot z| < 1$ , so  $B : \mathbb{H} \rightarrow \mathcal{D}$ , the unit disk. What is  $B \cdot 0$ ? What is  $B \cdot i\infty$ , i.e.  $\lim_{y \rightarrow \infty} B \cdot iy$ ? Show that  $B$  is a bijection by finding an inverse for  $B$ . (*Hint*: Solve  $w = \frac{z-i}{z+i}$  for  $z$ .) Let  $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . Show that  $BSB^{-1} : \mathcal{D} \rightarrow \mathcal{D}$  is a rotation by  $\pi$  around the origin.

**Numericals with  $SL(2, \mathbb{Z})$**

- P2. Set  $SL(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) : a, b, c, d \in \mathbb{Z} \right\}$ . Show that  $SL(2, \mathbb{Z})$  is a subgroup of  $SL(2, \mathbb{R})$ . Show that the action of  $SL(2, \mathbb{R})$  on  $\mathbb{H}$  is transitive, but the action of  $SL(2, \mathbb{Z})$  is not transitive
- P3. Let  $\mathcal{F} \subset \mathbb{H}$  be defined by  $\mathcal{F} = \{z \in \mathbb{C} : |z| \geq 1, |\operatorname{Re}(z)| \leq \frac{1}{2}\}$ . Draw a picture of  $\mathcal{F}$ . Define  $S, T \in SL(2, \mathbb{Z})$  by  $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . Which linear fractional transformations correspond to  $S$  and  $T$ ? What is the order of  $S$  as a matrix? What is the order of  $S$  as a linear fractional transformation? What does  $S(\mathcal{F})$  look like? What does  $T(\mathcal{F})$  look like? What about  $S^{-1}(\mathcal{F}), T^{-1}(\mathcal{F}), ST(\mathcal{F}), TS(\mathcal{F}), S^{17}T^{-3}ST^{44}(\mathcal{F})$ ?
- P4. Let  $G$  be the subgroup of  $SL(2, \mathbb{Z})$  generated by  $S$  and  $T$ , so elements of  $G$  are given by strings in  $S^{\pm 1}, T^{\pm 1}$ . Find some obvious relations (swear words) in  $G$ . (*Hint*: What is  $S^2$ ? Is  $ST = TS$ ?) If  $G'$  is the subgroup of the group of linear fractional transformations generated by  $S$  and  $T$ , what do these relations look like? Which group is easier to work with? For  $z = 4 + \frac{i}{2}$ , find  $A \in SL(2, \mathbb{Z})$  such that  $A \cdot z$  is in  $\mathcal{F}$ . Do the same for  $z = \frac{7}{8} + \frac{4}{5}i$ . Can you write  $A$  as an element of  $G$ ?
- P5. Show that every element of  $G$  is in  $SL(2, \mathbb{Z})$ . Write  $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$  as an element of  $G$  – i.e. as a word in  $S^{\pm 1}, T^{\pm 1}$ . (*Hint*: Pick  $z \in \mathcal{F}$  and figure out what combination of  $S$  and  $T$  takes  $A \cdot z$  back into  $\mathcal{F}$ .) Do the same for  $B = \begin{pmatrix} 9 & 7 \\ 5 & 4 \end{pmatrix}$ . Any interesting number theory involved?
- P6. Find a word  $W$  in  $T^{\pm 1}, S^{\pm 1}$  such that  $W = \begin{pmatrix} 4 & 5 \\ * & * \end{pmatrix}$ , i.e. with top row 4 5. Do the same for  $\begin{pmatrix} 9 & 7 \\ * & * \end{pmatrix}$  and  $\begin{pmatrix} 37 & -19 \\ * & * \end{pmatrix}$ . Since these matrices all have determinant one, you've solved equations like  $37d + 19c = 1$  for integers  $d, c$ . Can you systemize your search for  $W$ ? Is your algorithm different from Euclid's algorithm, or is it essentially the same?

**Group Actions**

- P7. Let a finite group  $G$  act transitively on a set  $X$ , and let  $S_x$  be the stabilizer subgroup of some  $x \in X$ . Check that  $|X| < \infty$ . Show that  $|G| = |X| \cdot |S_x|$ .
- P8. Check your answers to Set 10, P1 and P2 using the previous problem.

- P9. Compute the order of the symmetry groups of the octahedron, icosahedron and dodecahedron.

### Paradoxical Decompositions

- P10. Let  $G$  act via isometries on a set  $X$  with a distance function. Let  $A, B \subset X$ . Prove that the relation  $A \sim_G B$  iff  $A$  and  $B$  are  $G$ -equidecomposable is an equivalence relation on subsets of  $X$ .
- P11. PODASIP: Let  $A, B$  be rectangles in  $\mathbb{R}^2$  with the same area. We can chop up  $A$  into a finite number of rectangles and move the rectangles by isometries to exactly fill out  $B$ .
- P12. Let  $O(n+1)$  be the group of isometries of  $\mathbb{R}^{n+1}$  fixing the origin. Show that  $S^n$  is  $O(n)$ -paradoxical for  $n \geq 2$ . *Hint:* We will do the case  $n = 2$  in class using the existence of an injective homomorphism  $\phi : F_2 \rightarrow \text{SO}(3) \subset O(3)$ . Find an injective homomorphism  $\phi' : \text{SO}(3) \rightarrow O(n+1)$ . (Think of  $\text{SO}(2)$  sitting inside of  $O(3)$  as the rotations of the  $xy$ -plane.) This gives a copy of  $F_2$  inside  $O(n+1)$ , and we can repeat the proof from class. (The proof works for  $\text{SO}(n+1)$ , but it's a pain to describe orientation preserving/reversing transformations in  $\mathbb{R}^{n+1}$ .)
- P13. Show that  $\mathbb{R}^n \setminus \{0\}$  is  $\text{SO}(n)$ -paradoxical for  $n \geq 3$ . *Hint:* By P12, there are sets  $A_1, \dots, A_n, B_1, \dots, B_m$  giving a paradoxical decomposition of  $S^{n-1}$ . Let  $A'_i$  be all the rays from the origin through points of  $A_i$  excluding the origin, and define  $B'_j$  similarly. Show that the  $A'_i, B'_j$  give a paradoxical decomposition of  $\mathbb{R}^n \setminus \{0\}$ .
- P14. Show that  $\mathbb{R}^n$  is  $\text{Isom}(\mathbb{R}^n)$ -paradoxical for  $n \geq 3$ . *Hint:* We want to mimic the proof that  $S^2 \setminus D \sim_{\text{SO}(3)} S^2$  for a countable set  $D$  to show that  $\mathbb{R}^n \setminus \{0\} \sim_{\text{Isom}(\mathbb{R}^n)} \mathbb{R}^n$ . Consider rotations of the one point set  $\{0\}$  around a line not through the origin through powers of a “bad” angle..

### Linear Fractional Transformations and $\mathbb{CP}^1$

- P15. We can define a “linear transformation” on  $\mathbb{CP}^1$  by  $A([z, w]) = [az + bw, cz + dw]$  for  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  with all entries complex numbers. (The coordinates of  $A([z, w])$  are linear functions of  $z, w$ , hence the name.) Consider  $\mathbb{H}$  as a subset of  $\mathbb{C}$ , the set of finite points of  $\mathbb{CP}^1$ . Then for  $z \in \mathbb{H}$ ,  $A$  determines two actions on  $z$ , the linear fractional transformation  $A \cdot z = \frac{az+b}{cz+d}$  and  $A([z, 1])$ . Show that these two transformations are the same. So linear fractional transformations come from linear transformations of  $\mathbb{CP}^1$ .
- P16. For  $A \in \text{SL}(2, \mathbb{R})$ , set  $A \cdot i\infty = \lim_{y \rightarrow \infty} A \cdot iy$ . Since  $i\infty$  also corresponds to  $[1, 0] \in \mathbb{CP}^1$ , we can also set  $A \cdot i\infty = A([1, 0])$ . Show that these two definitions agree. Define  $S, T \in \text{SL}(2, \mathbb{Z})$  as in P3. What is  $S(i\infty)$ ? What is  $T(i\infty), ST(i\infty), TS(i\infty), T^{-7}ST^2S(i\infty)$ ? What are the possible values of  $W(i\infty)$  for  $W \in G$ , the group in P4? Can you find a word  $W$  such that  $W(i\infty) = 943/19$ ? What is  $\lim_{n \rightarrow \infty} (TS)^n(i\infty)$ ?