Exploration – The Upper Half Plane and the Unit Disk

P1. Let \( B = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \) be the complex linear fractional transformation \( B \cdot z = \frac{z - i}{z + i} \). Show that if \( z \in \mathbb{H} \), then \(|B \cdot z| < 1 \), so \( B : \mathbb{H} \to \mathbb{D} \), the unit disk. What is \( B \cdot 0 \)? What is \( B \cdot i\infty \), i.e. \( \lim_{y \to \infty} B \cdot iy \)? Show that \( B \) is a bijection by finding an inverse for \( B \). (Hint: Solve \( w = \frac{z - i}{z + i} \) for \( z \)). Let \( S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). Show that \( BSB^{-1} : \mathbb{D} \to \mathbb{D} \) is a rotation by \( \pi \) around the origin.

Numericals with \( \text{SL}(2, \mathbb{Z}) \)

P2. Set \( \text{SL}(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}) : a, b, c, d \in \mathbb{Z} \right\} \). Show that \( \text{SL}(2, \mathbb{Z}) \) is a subgroup of \( \text{SL}(2, \mathbb{R}) \). Show that the action of \( \text{SL}(2, \mathbb{R}) \) on \( \mathbb{H} \) is transitive, but the action of \( \text{SL}(2, \mathbb{Z}) \) is not transitive.

P3. Let \( \mathcal{F} \subset \mathbb{H} \) be defined by \( \mathcal{F} = \{ z \in \mathbb{C} : |z| \geq 1, |\text{Re}(z)| \leq \frac{1}{4} \} \). Draw a picture of \( \mathcal{F} \). Define \( S, T \in \text{SL}(2, \mathbb{Z}) \) by \( S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), \( T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). Which linear fractional transformations correspond to \( S \) and \( T \)? What is the order of \( S \) as a matrix? What is the order of \( S \) as a linear fractional transformation? What does \( S(F) \) look like? What does \( T(F) \) look like? What about \( S^{-1}(F), T^{-1}(F), ST(F), STS(F), S^{17}T^{-1}ST^{14}(F) \)?

P4. Let \( G \) be the subgroup of \( \text{SL}(2, \mathbb{Z}) \) generated by \( S \) and \( T \), so elements of \( G \) are given by strings in \( S^{\pm 1}, T^{\pm 1} \). Find some obvious relations (swear words) in \( G \). (Hint: What is \( S^2 \)? Is \( ST = TS \)?) If \( G' \) is the subgroup of the group of linear fractional transformations generated by \( S \) and \( T \), what do these relations look like? Which group is easier to work with? For \( z = 4 + \frac{i}{2} \), find \( A \in \text{SL}(2, \mathbb{Z}) \) such that \( A \cdot z \) is in \( \mathcal{F} \). Do the same for \( z = \frac{7}{8} + \frac{1}{8} i \). Can you write \( A \) as an element of \( G' \)?

P5. Show that every element of \( G \) is in \( \text{SL}(2, \mathbb{Z}) \). Write \( A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \) as an element of \( G \) – i.e. as a word in \( S^{\pm 1}, T^{\pm 1} \). (Hint: Pick \( z \in \mathcal{F} \) and figure out what combination of \( S \) and \( T \) takes \( A \cdot z \) back into \( \mathcal{F} \).) Do the same for \( B = \begin{pmatrix} 9 & 7 \\ 5 & 4 \end{pmatrix} \) Any interesting number theory involved?

P6. Find a word \( W \) in \( T^{\pm 1}, S^{\pm 1} \) such that \( W = \begin{pmatrix} 4 & 5 \\ * & * \end{pmatrix} \), i.e. with top row 4 5. Do the same for \( \begin{pmatrix} 9 & 7 \\ * & * \end{pmatrix} \) and \( \begin{pmatrix} 37 & -19 \\ * & * \end{pmatrix} \). Since these matrices all have determinant one, you’ve solved equations like \( 37d + 19c = 1 \) for integers \( d, c \). Can you systemize your search for \( W \)? Is your algorithm different from Euclid’s algorithm, or is it essentially the same?

Group Actions

P7. Let a finite group \( G \) act transitively on a set \( X \), and let \( S_x \) be the stabilizer subgroup of some \( x \in X \). Check that \(|X| < \infty \). Show that \(|G| = |X| \cdot |S_x| \).

P8. Check your answers to Set 10, P1 and P2 using the previous problem.
P9. Compute the order of the symmetry groups of the octahedron, icosahedron and dodecahedron.

Paradoxical Decompositions

P10. Let $G$ act via isometries on a set $X$ with a distance function. Let $A, B \subset X$. Prove that the relation $A \sim_G B$ iff $A$ and $B$ are $G$-equidecomposable is an equivalence relation on subsets of $X$.

P11. PODASIP: Let $A, B$ be rectangles in $\mathbb{R}^2$ with the same area. We can chop up $A$ into a finite number of rectangles and move the rectangles by isometries to exactly fill out $B$.

P12. Let $O(n+1)$ be the group of isometries of $\mathbb{R}^{n+1}$ fixing the origin. Show that $S^n$ is $O(n)$-paradoxical for $n \geq 2$. Hint: We did the case $n = 2$ in class using the existence of an injective homomorphism $\phi : F_2 \to SO(3) \subset O(3)$. Find an injective homomorphism $\phi' : SO(3) \to O(n + 1)$. (Think of $SO(2)$ sitting inside of $O(3)$ as the rotations of the $xy$-plane.) This gives a copy of $F_2$ inside $O(n+1)$, and we can repeat the proof from class. (The proof works for $SO(n+1)$, but it’s a pain to describe orientation preserving/reversing transformations in $\mathbb{R}^{n+1}$.)

P13. Show that $\mathbb{R}^n \setminus \{0\}$ is $SO(n)$-paradoxical for $n \geq 3$. Hint: By P12, there are sets $A_1, \ldots, A_n, B_1, \ldots, B_m$ giving a paradoxical decomposition of $S^{n-1}$. Let $A'_i$ be all the rays from the origin through points of $A_i$ excluding the origin, and define $B'_j$ similarly. Show that the $A'_i, B'_j$ give a paradoxical decomposition of $\mathbb{R}^n \setminus \{0\}$.

P14. Show that $\mathbb{R}^n$ is $\text{Isom}(\mathbb{R}^n)$-paradoxical for $n \geq 3$. Hint: We want to mimic the proof that $S^2 \setminus D \sim_{SO(3)} S^2$ for a countable set $D$ to show that $\mathbb{R}^n \setminus \{0\} \sim_{\text{Isom}(\mathbb{R}^n)} \mathbb{R}^n$. Consider rotations of the one point set $\{0\}$ around a line not through the origin through powers of a “bad” angle.

Linear Fractional Transformations and $\mathbb{C}P^1$

P15. We can define a “linear transformation” on $\mathbb{C}P^1$ by $A([z, w]) = [az + bw, cz + dw]$ for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with all entries complex numbers. (The coordinates of $A([z, w])$ are linear functions of $z, w$, hence the name.) Consider $\mathbb{H}$ as a subset of $\mathbb{C}$, the set of finite points of $\mathbb{C}P^1$. Then for $z \in \mathbb{H}$, $A$ determines two actions on $z$, the linear fractional transformation $A \cdot z = \frac{az + b}{cz + d}$ and $A([z, 1])$. Show that these two transformations are the same. So linear fractional transformations come from linear transformations of $\mathbb{C}P^1$.

P16. For $A \in \text{SL}(2, \mathbb{R})$, set $A \cdot i\infty = \lim_{y \to \infty} A \cdot iy$. Since $i\infty$ also corresponds to $[1, 0] \in \mathbb{C}P^1$, we can also set $A \cdot i\infty = A([1, 0])$. Show that these two definitions agree. Define $S, T \in \text{SL}(2, \mathbb{Z})$ as in P3. What is $S(i\infty)$? What is $T(i\infty)$, $ST(i\infty)$, $TS(i\infty)$, $T^{-7}ST^2S(i\infty)$? What are the possible values of $W(i\infty)$ for $W \in G$, the group in P4? Can you find a word $W$ such that $W(i\infty) = 943/19$? What is $\lim_{n \to \infty} (TS)^n(i\infty)$?