Geometry and Symmetry, Problem Set 12
Summer 2009

Asteroids. Exploration.

P1. In the arcade classic Asteroids, the (x, y)-coordinates are each taken mod 1, so your screen is $\{(x, y)|0 \le x, y \le 1\}$, and an asteroid or rocket moving off the screen at angle θ at e.g. $(0, .537)$ reappears on the screen at $(1, .537)$ moving at the same angle θ. (Here we are assuming the screen is $[0, 1] \times [0, 1]$.) So this is "billiards with wrapping around instead of reflecting."

Start your rocket at a point on the screen moving at some angle θ. For which points and angles does your rocket trajectory form a closed loop? If your trajectory does not close up, how much of the screen does it cover – 20%, 99%?

Now fix θ. Show that $\mathbb{R}$ acts on the screen by setting $t \cdot (x, y)$ to be where your rocket, starting at (x, y), ends up at time t. (Rockets travel at unit speed, and we move them into the past in the obvious way.) Find a slice for this action. How does the slice depend upon θ?

Number theory and linear algebra–a digression.

Recall that the Fibonacci numbers are given by the recursion formula $F_k = F_{k-1}+F_{k-2}, F_0 = F_1 = 1$, that $(1 + \sqrt{5})/2$ has continued fraction expansion $[\dot{1}]$, and that the convergents are $1/1, 2/1, 3/2, 5/3, \ldots, F_k/F_{k-1}, \ldots$.

P2. Show that the Fibonacci numbers satisfy

$$\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}F_{k-1}\\ F_{k-2}\end{pmatrix} = \begin{pmatrix}F_k\\ F_{k-1}\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^{k-1}\begin{pmatrix}1\\ 1\end{pmatrix}$$

P3. Let $A : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation. Recall that a vector $v \in \mathbf{R}^2$ is an *eigenvector* of A with *eigenvalue* λ if $Av = \lambda v$, and that λ is an eigenvalue iff $\det(A - \lambda \cdot \mathrm{Id}) = 0$. For the matrix $A = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, show that the eigenvalues are $(1 \pm \sqrt{5})/2$, with corresponding eigenvectors

$$v_1 = \begin{pmatrix}\frac{1+\sqrt{5}}{2}\\ 1\end{pmatrix}, v_2 = \begin{pmatrix}\frac{1-\sqrt{5}}{2}\\ 1\end{pmatrix}.$$

(Note: if v is an eigenvector, so is $c \cdot v$ for any $c \in \mathbf{R}$. Here we arbitrarily choose the y component of the eigenvectors to be 1.)

P4. Show that

$$\begin{pmatrix}1\\ 1\end{pmatrix} = \left(\frac{5+\sqrt{5}}{10}\right)v_1 + \left(\frac{5-\sqrt{5}}{10}\right)v_2.$$

Conclude that

$$\begin{aligned}\begin{pmatrix}F_k\\ F_{k-1}\end{pmatrix} &= \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^{k-1}\left[\left(\frac{5+\sqrt{5}}{10}\right)v_1 + \left(\frac{5-\sqrt{5}}{10}\right)v_2\right]\\ &= \frac{5+\sqrt{5}}{10}\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^{k-1}v_1 + \frac{5-\sqrt{5}}{10}\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^{k-1}v_2\\ &= \left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}v_1 + \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}v_2.\end{aligned}$$

Thus

$$F_k = \left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^k,$$

an amazing formula.

Amenable groups.

Recall that a group G is *amenable* if there exists a G-invariant, finitely additive, finite measure on G; i.e. a function $\mu : \mathcal{P}(G) \to [0, \infty)$ such that

(i) $\mu(g \cdot A) = \mu(A)$, for all $A \subset G$;

(ii) $\mu(A_1 \cup \ldots \cup A_n) = \sum_i \mu(A_i)$ for pairwise disjoint subsets $A_i \subset G$.

Here $\mathcal{P}(G) = \{X | X \subset G\}$ is the power set of G. We always assume that $\mu(G) = 1$ to rule out the stupid measure.

P5. Let $B(G)$ be the set of real valued bounded functions on G: $B(G) = \{f : G \to \mathbb{R} | \exists M > 0 \text{ with } |f(g)| < M, \forall g \in G\}$. A a *left-invariant mean* on G is a function $\int_G : B(G) \to \mathbf{R}$ with $\int_G af + bg = a\int_G + b\int_G g$, $\int_G f = \int_g f(g^{-1}\cdot)$, $\inf(f) \leq \int_G f \leq \sup(f))$. Show that G is amenable iff G has a left-invariant mean. *Hint: If G has a mean, set $\mu(A) = \int_G \chi_A$, where the* characteristic function *$\chi_A(x)$ equals one if $x \in A$ and equals zero if $x \notin A$. For the converse, first define $\int_G$ on step functions f, functions with range a finite set $\{r_1, \ldots, r_k\}$, by $\int_G f = \sum_i r_i \mu(f^{-1}(r_i))$ (draw a picture). Then approximate an arbitrary function on G by a sequence of step functions and take a limit, checking that the limit is independent of the approximating sequence.*

P6. Show that $\mathbf{Z}$ is amenable. Warning: you must be comfortable with the Hahn-Banach theorem, which states that if V is a subspace of a real vector space X, if $p : X \to \mathbf{R}$ satisfies $p(x + y) \leq p(x) + p(y), p(c \cdot x) = c \cdot p(x)$ for $x, y \in X, c \geq 0$, and if $\lambda : V \to \mathbf{R}$ is linear with $\lambda(x) \leq p(x)$ for $x \in V$, then there exists a linear function $\Lambda : X \to \mathbf{R}$ with $\Lambda|_V = \lambda$ and $-p(-x) \leq \Lambda(x) \leq p(x)$. See e.g. Rudin's *Functional Analysis*, Ch. 3. *The proof of the Hahn-Banach theorem uses our old friend, the Axiom of Choice.* Now set $V = \{f \in B(\mathbf{Z}) : \lim_{n\to\infty}(1/n)(f(-n) + f(-n+1) + \ldots + f(n-1) + f(n)) \text{ exists}\}$, set $\lambda : V \to \mathbf{R}$ by $\lambda(f) = \lim_{n\to\infty}(1/n)(f(-n) + f(-n+1) + \ldots + f(n-1) + f(n))$, and apply Hahn-Banach. Note: This gives a method to assign a average value to every double-sided bounded infinite sequence of real numbers. We need the Axiom of Choice because there is no way to list or describe the sequences outside of V – you try. This problem from the 1920s was the motivation for defining amenable groups.

P7. Show that if G is amenable and $f : G \to H$ is a homomorphism, then H is amenable. *Hint: set $\mu_H(A) = \mu_G(f^{-1}(A))$.* Use the First Isomorphism theorem to conclude that if G is amenable and H is a normal subgroup of G, then G/H is amenable.

P8. Show that if H is a subgroup of an amenable group G, then H is amenable. *Hint: Let $\mathcal{S}$ consist of one representative of each equivalence class in G/H; i.e. $\mathcal{S}$ is a slice for the action of H on G. For $f \in B(H)$, define $\bar{f} \in B(G)$ by $\bar{f}(g) = f(h)$ if $g \in [h]$. Check that $\bar{f}$ is well defined. Now set $\int_H f = \int_G \bar{f}$.* Conclude as in class that a group containing F_2 is nonamenable.

P9. Show that if N is an amenable, normal subgroup of G with G/N amenable, then G is amenable. *Hint: Take $f \in B(G)$. For $x, y \in G$, if $[x] = [y] \in G/N$, then $x = n'y$ for some $n' \in N$ and $(x \cdot f)(n) = ((n'y \cdot f)(n) = (n' \cdot (y \cdot f)(n)$ for all $n \in N$. (Why?) Show that*

$\int_N((x \cdot f|_N) = \int_N(y \cdot f|_N)$. *Thus* $F(x) \equiv \int_N(x \cdot f|_N)$ *is an element of* $B(G/N)$. *Now set* $\int_G f = \int_{G/N} F$.

P10. Show that if G, H are amenable, then $G \times H$ is amenable. *Hint: Show that* G *is normal in* $G \times H$ *and use the previous problem.*

P11. Show that all finitely presented abelian groups are amenable. *Hint: Use/look up the result that any finitely presented abelian group is isomorphic to* $\mathbf{Z}^k \times \mathbf{Z}_{p_1^{\alpha_1}} \times \ldots \times \mathbf{Z}_{p_s^{\alpha_s}}$ *for some primes* $p_1, \ldots, p_s$. Conclude that any abelian group is amenable. *Hint: The previous hint applies to abelian groups with a finite number of generators and possibly infinitely many relations. For any abelian group* G*, find normal subgroups* G_α *with a finite number of generators, and with* G *the limit of the* G_α *in an appropriate sense. This will probably take some reading in e.g. Lang's "Algebra" on your part.* What does the finite measure μ look like on $(\mathbb{R}, +)$?

Solvable groups.

P12. Show that the commutator subgroup $[G, G]$ is normal in G. Show that $G/[G, G]$ is always abelian. *Hint: if not, then there is a commutator outside* $[G, G]$*, a contradiction.*

P13. Set $G^{(1)} = [G, G], G^{(2)} = [G^{(1)}, G^{(1)}]$, etc. We say that G is *(k-step) solvable* if $G^{(k)} = \{e\}, G^{(k-1)} \neq \{e\}$. Show that F_2 is not solvable.

P14. Set $G^1 = [G, G], G^2 = [G, G^1]$, etc. We say that G is *(k-step) nilpotent* if $G^k = \{e\}, G^{k-1} \neq \{e\}$. Show that every abelian group is 1-step nilpotent, and that every nilpotent group is solvable. Show that the group $G = \langle a, b, c\rangle / [a, b] = b, [a, c] = 1, [b, c] = 1$ is solvable but not nilpotent. *Hint: Show that* $[G, G] \subset \langle b \rangle$*, and so* $G^{(2)} = \{e\}$*. Thus* G *is solvable. To show that* G *is not nilpotent, prove by induction that* $b \in G^k$ *for all* k*. Subhint: since* $[a, b] = b$*,* b *is in* $[G, G]$*. Then* $[b, a] \in G^2$*. Since the inverse of* $[b, a]$ *is* $[a, b]$*,* $[a, b]$ *is also in* G^2*. Since* $[a, b] = b$*, we get* $b \in G^2$*. Now continue.*

P15. Is the isometry group of the equilateral triangle solvable? Is it nilpotent? What about the isometry group of the square? What about the isometry groups of the Platonic solids? Note: Galois' famous proof of the non-solvability of the fifth degree polynomial equation depends upon the fact that S_5 is non-solvable!

P16. Show that $G^{(k)}$ is a normal subgroup of $G^{(k-1)}$.

P17. Show that solvable groups are amenable. *Hint: use induction on* k*, P9, P12, P16.*

P18. Show that finitely presented nilpotent groups have polynomial growth, i.e. there exist a constant $C > 0$ and $n \in \mathbb{Z}^+$ such that the $N(k)$, the number of words of length at most k, satisfies $N(k) \leq Ck^n$. *Hint: induction on* k*, where the group is* k*-step nilpotent.* It is a famous theorem of Gromov that the converse is (almost) correct. Why does your proof break down if G is solvable but not nilpotent?