Geometry and Symmetry, Problem Set #2
Summer 2009

Exploration

P1. Four children take walks every day, rain or shine. They walk in two rows of two children each, but they like to have new partners every day. What is the maximum number of days the children can take walks such that no child has the same partner on two different days? What if there are nine children who like to walk in rows of three? What if there are six children who like to walk in rows of two? What if there are six children who like to walk in rows of three?

Lines and hyperplanes in $\mathbb{R}^n$

$\mathbb{R}^n$, n-dimensional Euclidean space, is by definition $\{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, i = 1, 2, \ldots, n\}$.

P2. What is the vector equation of the line joining $(1, 2, 3, 4)$ and $(-2, 3, -5, 7)$ in $\mathbb{R}^4$? Is it easy/possible/worthwhile to describe this line in coordinates?

P3. Find a point (a, b, c, d, e) and vectors $\vec{v}, \vec{w} \in \mathbb{R}^5$ such that the plane containing $(1, 2, -1, 3, 2)$, $(2, 3, 4, 1, 1)$, $(-3, 4, 1, -2, -2)$ is given by the set of vectors $(a, b, c, d, e) + \lambda\vec{v} + \mu\vec{w}$, for $\lambda, \mu \in \mathbb{R}$.

P4. Describe the line joining $\vec{v} = (v_1, \ldots, v_n)$ and $\vec{w} = (w_1, \ldots, w_n)$ in $\mathbb{R}^n = \{(x^1, \ldots, x^n) : x^i \in \mathbb{R}\}$. Describe the plane containing $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$, $(z_1, \ldots, z_n)$.

*P5. What is the equation of the "hyperplane" H_1 in $\mathbb{R}^4 = \{(x^1, \ldots, x^4) : x^i \in \mathbb{R}\}$ containing $(0, 0, 0, 0)$, $(3, -2, 4, 4)$, $(0, 3, 6, -3)$, $(1, 2, 8, 0)$? The hyperplane H_2 containing $(0, 0, 0, 1)$, $(2, 0, 0, 0)$, $(1, 1, 2, 1)$, $(1, 1, 4, -1)$? The hyperplane H_3 containing $(0, 0, 0, 0)$, $(1, -2, 0, 0)$, $(0, 0, -1, -1)$, $(1, 0, 0, 2)$? Write $H_1 \cap H_3$ as the plane $\vec{v} + \lambda\vec{w} + \mu\vec{u}$ for vectors $\vec{v}, \vec{w}, \vec{u} \in \mathbb{R}^4$. *Hint:* You can take $\vec{v} = \vec{0}$. (Why?) Do the same for $H_1 \cap H_2$. *Hint:* First find a plane H_2' parallel to H_2 and passing through the origin. Then find $\vec{w}, \vec{u}$ for $H_1 \cap H_2'$. Now find a vector $\vec{v}$ lying in both H_1, H_2. To do this, assume that the intersection $H_1 \cap H_3$ is a plane (why?), so a little dangerously assume we can shose the x-coordinate of $\vec{v}$ to be e.g. 1. With this choice of x, find an equation for $\vec{v}$ – the equation will still involve making another choice of, say w coordinate of $\vec{v}$. Continue until you find a choice for $\vec{v}$. Finally, show that $\vec{v}, \vec{w}, \vec{u}$ are the vectors we want.

Lines in $\mathbb{C}^n$ and $\mathbb{Z}_k^n$

P6. Set $\mathbb{Z}_3^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{Z}_3, i = 1, \ldots, n\}$. For $p_1 = (1, 2, 0, 2)$, $p_2 = (0, 0, 2, 2) \in \mathbb{Z}_3^4$, why is $\{(1, 2, 0, 2) + \lambda(2, 1, 2, 0) : \lambda \in \mathbb{R}\}$ a *bad* definition of the line joining p_1 and p_2? What is a good definition of this line? Are the lines $x + y = 2$ and $2x + 2y = 4$ the same lines in $\mathbb{Z}_3^2$? In $\mathbb{Z}_4^2$?

P7. What is $\mathbf{C}^n$? What is (a good definition of) the line joining $(1, 2, 3, \ldots, n)$ and $(1+2i, 2+3i, \ldots, n+(n+1)i)$? What is the line joining two vectors $\vec{v}, \vec{w} \in \mathbf{C}^n$?

Numericals

P8. Find a vector $\vec{v} \in \mathbb{R}^3$ such that the intersection of the planes $2x + 3y - z = 0$ and $x + y + 4z = 0$ in $\mathbb{R}^3$ is the line $\ell = \{\lambda\vec{v} : \lambda \in \mathbb{R}\}$.

P9. Find a vector $\vec{v} \in \mathbb{C}^3$ such that the intersection of the planes $(2+i)x + (3+2i)y - z = 0$ and $(1+i)x + y - (2-3i)z = 0$ in $\mathbb{C}^3$ is the line $\ell = \{\lambda\vec{v} : \lambda \in \mathbb{C}\}$.

P10. Find a vector $\vec{v} \in \mathbb{Z}_5^3$ (i.e. $(\mathbb{Z}_5)^3$) such that the intersection of the planes $2x + 3y - z = 0$ and $x + y - 3z = 0$ in $\mathbb{Z}_5^3$ is the line $\ell = \{\lambda\vec{v} : \lambda \in \mathbb{Z}_5\}$.

P11. Write down (and justify) an example of two parallel lines in $\mathbb{R}^2$. Write down an example in $\mathbb{C}^2$, in $\mathbb{Z}_5^2$, in $\mathbb{R}^3$, in $\mathbb{C}^3$, in $\mathbb{Z}_5^3$.

P12. PODASIP: The equations $ax + by = c$ and $dx + ey = f$ in $\mathbb{R}^2$ describe the same line iff there exists $\lambda \in \mathbb{R}$ with $a = \lambda d, b = \lambda e, c = \lambda f$. What if we replace $\mathbb{R}$ by $\mathbb{C}, \mathbb{Z}_5$ or $\mathbb{Z}_4$?

The 7 Point Geometry

In class we wrote down a "geometry" with seven points and seven lines, with each line consisting of three points. This geometry had the properties (i) two distinct distinct points determine a unique line, and (ii) every pair of distinct lines intersect in a unique point. Let the set of points be G. Recall that a symmetry of G is a bijection of G taking lines to lines.

P13. Given any two lines ℓ_1, ℓ_2 in the seven point geometry, is there a symmetry taking ℓ_1 to ℓ_2? Is this symmetry unique? (Don't forget the case where $\ell_1 = \ell_2$.) Given two points a, b in G, is there a symmetry taking a to b? Is this symmetry unique?

P14. How many symmetries fix a point? How many symmetries fix a line? How many fix two lines?

P15. Does there exist a geometry with seven points and properties (i) and (ii) with fewer than seven lines? (We require our geometries to have four points, no three of which are collinear to exclude trivial examples like one line with seven points.)

P16. Does there exist a geomety with properties (i) and (ii) with two points? With three points? With four or five or six points? (Exclude trivial cases as in P14.)

Synthetic Geometry – Food for Thought

A (synthetic) geometry is a set $\mathbf{G}$, called the *points* of the geometry, together with a collection $\mathcal{L}_\mathbf{G}$ of subsets of $\mathbf{G}$, called the *lines* of the geometry. We use the standard language of geometry and say that a point $P \in \mathbf{G}$ lies on a line $\ell \in \mathcal{L}_\mathbf{G}$ or that the line ℓ passes through P if $P \in \ell$. Similarly, a set C of points in $\mathbf{G}$ are *collinear* if there is a line $\ell \in \mathcal{L}_\mathbf{G}$ such that $C \subseteq \ell$, and two lines ℓ_1, ℓ_2 are *parallel* if ℓ_1 and ℓ_2 have no points in common. Notice that we don't have notions for distance, angle, area, etc. from ordinary Euclidean geometry.

For each of the geometries below, which of the following properties hold?

A. There is a unique line containing any pair of distinct points.

B. Any two lines intersect at a unique point.

C. Given a line ℓ_1 and a point P not on ℓ_1, there exists a unique line ℓ_2 through P parallel to ℓ_1.

P17. $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^3$, where the lines are the usual lines.

P18. $\mathbb{C}^1$, $\mathbb{C}^2$, $\mathbb{C}^3$, where you defined the lines in P7.

P19. $\mathbb{Z}_3^2$, $\mathbb{Z}_3^3$, where you defined the lines in P6.

P20. S^1, S^2, S^3, where the points of S^n are the elements of $\{(x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} : (x^1)^2 + \ldots + (x^{n+1})^2 = 1\}$, and lines are great circles/equators, the intersections of (hyper)planes through the origin with $\mathbf{S}^n$. (A hyperplane through the origin in $\mathbb{R}^{n+1}$ is the solution set to $a_1x^1 + \ldots + a_{n+1}x^{n+1} = 0$.)

P21. $(x, y) \in \mathbb{R}^2$ with $y > 0$, and the lines are either (a) vertical lines $\ell = \{(x_0, y) : y > 0\}$ for fixed x_0, or (b) semicircles which hit the real axis at right angles.

P22. The set of points is $\{a, b, c, d, e, f, g\}$ and the set of lines is

$$\{\{a, b, c\}, \{c, d, e\}, \{a, e, f\}, \{a, g, d\}, \{c, g, f\}, \{b, g, e\}, \{b, d, f\}\}.$$

I Hate Checking Associativity

P23. Let X, Y, Z, W be sets and let $f : X \to Y, g : Y \to Z, h : Z \to W$ be functions. Show that $h \circ (g \circ f) = (h \circ g) \circ f$ as functions from X to W. Conclude that the set of symmetries of any (reasonable) geometry is associative.