Geometry and Symmetry, Problem Set #3
Summer 2009

Exploration. Affine and Projective Geometry.

A geometry $\mathbf{G}$ is an *affine plane* if the following three properties are satisfied:

Af1. For every pair of distinct points of $\mathbf{G}$, there is a unique line $\ell \in \mathcal{L}_{\mathbf{G}}$ containing them.
Af2. There is a four element subset of $\mathbf{G}$ in which no three points are collinear.
Af3. For each line ℓ and point $P \notin \ell$, there is a unique line ℓ' on P parallel to ℓ.

A geometry $\mathbf{G}$ is a *projective plane* if $\mathbf{G}$ has the following three properties:

Pr1. For every pair of distinct points of $\mathbf{G}$, there is a unique line $\ell \in \mathcal{L}_{\mathbf{G}}$ containing them.
Pr2. There is a four element subset of $\mathbf{G}$ in which no three points are collinear.
Pr3. Any two lines in $\mathbf{G}$ intersect in at least one point.

P1. An affine plane has *order n* if every line has n points on it. Can you draw affine planes of order 2, 3, 4? Can you draw an affine plane with some lines containing 3 points and some lines containing 4 points?

P2. Redo P1 for projective planes.

P3. Show that $\mathbb{Z}_p^2$ is an affine plane for p prime, but that $\mathbb{Z}_p^3$ is neither affine nor projective. How many lines are there in $\mathbb{Z}_3^2$? How many lines are there in $\mathbb{Z}_p^2$? How many families of parallel lines are there in each case? How many lines are there in $\mathbb{Z}_p^n$?

Numericals

P4. In class we considered matrices of the form $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a^2 + b^2 = c^2 + d^2 = 1$ and $(a, c) \perp (b, d)$. Now drop these conditions on a, b, c, d and consider

$$A : \mathbb{R}^2 \to \mathbb{R}^2, \; A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

For $A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$, what is $A\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $A\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $A\begin{pmatrix} 7 \\ 0 \end{pmatrix}$, $A\begin{pmatrix} 0 \\ 8 \end{pmatrix}$, $A\begin{pmatrix} 7 \\ 8 \end{pmatrix}$? Do the same for A replaced by $B = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$, and by $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. What is special about I?

P5. For A as in P4, can you find $x, y \in \mathbb{R}$ such that $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$? For B as in P4, can you find $x, y \in \mathbb{R}$ such that $B\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$?

P6. For A, B as in P4, find all vectors (x, y) such that $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 12 \end{pmatrix}$. Find all vectors (x, y) such that $B\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 12 \end{pmatrix}$.

P7. Is A in P4 injective? Is B? Is A surjective? Is B?

P8. For the matrix A in P4, find vectors $(e, g), (f, h)$ such that $A\begin{pmatrix} e \\ g \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $A\begin{pmatrix} f \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Set $C = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. What is the composition $A \circ C$ as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$? What is the composition $C \circ A$?

P9. Consider the *shear* matrix $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Compute $S\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $S\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. What is the area of the image of the unit square under S? What is the area of the image of the rectangle with vertices $(0,0), (x,0), (0,y), (x,y)$? What is the area of the image of the unit circle? Is S an isometry? Can you quickly write down the matrix T such that $S \circ T = T \circ S = I$, where I is as in P4?

P10. Define the product of two matrices by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

For the matrices in P4, P8, P9, what is AB, BA, AC, CA, ST, TS. How do these products compare with the compositions $A \circ B, B \circ A$, etc.? (Here $A, B, A \circ B : \mathbb{R}^2 \to \mathbb{R}^2$ as in P4.)

P11. PODASIP: The matrix of translation by the vector $(2,3)$ is $\begin{pmatrix} 3 & 2 \\ 3 & 4 \end{pmatrix}$.

P12. PODASIP: The set of 2×2 matrices is a group under matrix multiplication.

P13. PODASIP: For any 2×2 matrix A and any vector $\begin{pmatrix} a \\ b \end{pmatrix}$, the vectors

$$A\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad A\begin{pmatrix} a \\ 0 \end{pmatrix} \quad A\begin{pmatrix} 0 \\ b \end{pmatrix} \quad A\begin{pmatrix} a \\ b \end{pmatrix}$$

form the vertices of a rectangle.

Examples of Groups

P14. A *subgroup* H of a group G is a subset $H \subset G$ which is itself a group with respect to G's operation. Show that the even integers are a subgroup of $(\mathbb{Z}, +)$, but the odd integers are not. Find all the subgroups of $(\mathbb{Z}_n, +)$.

P15. Show that the complex numbers of length one are a subgroup of $\mathbb{C}^* = \mathbb{C} - \{0\}$, but that the nonzero complex numbers of length at most one are not. (The length of a complex number $z = x + iy$ is just its distance from the origin: $|z| = \sqrt{x^2 + y^2}$.)

P16. Let θ be a fixed angle and let R_θ be rotation by angle θ counterclockwise around the origin. Let R_θ^2 denote $R_\theta \circ R_\theta$. What is R_θ^2 geometrically? Define R_θ^n similarly for $n \in \mathbb{Z}^+$. Let R_θ^{-1} denote rotation by angle θ clockwise around the origin, $R_\theta^{-2} = R_\theta^{-1} \circ R_\theta^{-1}$, etc. Show that $\{R_\theta^n | n \in \mathbb{Z}\}$ is a group with respect to composition. For which angles is this group finite? For which angles is this group infinite?

P17. PODASIP: The set of rotations around a fixed point in $\mathbb{R}^2$ is a subgroup of the group of isometries of $\mathbb{R}^2$. The set of translations in $\mathbb{R}^2$ is a subgroup. The set of all rotations around all points in $\mathbb{R}^2$ is a subgroup. The set of all translations and all rotations around all points in $\mathbb{R}^2$ is a subgroup. The set of reflections across all lines is a subgroup.

P18. PODASIP: Let S be a finite set. The set $\mathcal{F}(S)$ of all functions $f : S \to S$ is a group with respect to composition of functions (i.e. the operation on $f, g \in \mathcal{F}(S)$ is $f \circ g$).

Miscellaneous

P19. Prove the triangle inequality: for $P, Q, R \in \mathbb{R}^2$, $d(P,Q) + d(Q,R) \le d(P,R)$ with equality iff P, Q, R are collinear with Q between P and R. (Be sure to define "between" precisely.)