Exploration. Affine and Projective Geometry.

A geometry $G$ is an affine plane if the following three properties are satisfied:

Af1. For every pair of distinct points of $G$, there is a unique line $\ell \in \mathcal{L}_G$ containing them.
Af2. There is a four element subset of $G$ in which no three points are collinear.
Af3. For each line $\ell$ and point $P \notin \ell$, there is a unique line $\ell' \parallel P$ on $P$ parallel to $\ell$.

A geometry $G$ is a projective plane if $G$ has the following three properties:

Pr1. For every pair of distinct points of $G$, there is a unique line $\ell \in \mathcal{L}_G$ containing them.
Pr2. There is a four element subset of $G$ in which no three points are collinear.
Pr3. Any two lines in $G$ intersect in at least one point.

P1. An affine plane has order $n$ if every line has $n$ points on it. Can you draw affine planes of order 2, 3, 4? Can you draw an affine plane with some lines containing 3 points and some lines containing 4 points?

P2. Redo P1 for projective planes.

P3. Show that $\mathbb{Z}_p^2$ is an affine plane for $p$ prime, but that $\mathbb{Z}_p^3$ is neither affine nor projective. How many lines are there in $\mathbb{Z}_p^2$? How many lines are there in $\mathbb{Z}_p^3$? How many families of parallel lines are there in each case? How many lines are there in $\mathbb{Z}_p^n$?

Numericals

P4. In class we will consider matrices of the form $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a^2 + b^2 = c^2 + d^2 = 1$ and $(a, c) \perp (b, d)$. Now drop these conditions on $a, b, c, d$ and consider

$$T_A : \mathbb{R}^2 \to \mathbb{R}^2, \quad T_A \left( \begin{array}{l} x \\ y \end{array} \right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

For $A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$, what is $T_A \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $T_A \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $T_A \begin{pmatrix} 7 \\ 0 \end{pmatrix}$, $T_A \begin{pmatrix} 0 \\ 8 \end{pmatrix}$, $T_A \begin{pmatrix} 7 \\ 8 \end{pmatrix}$?

Do the same for $A$ replaced by $B = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$, and by $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. What is special about $I$?

P5. For $A$ as in P4, can you find $x, y \in \mathbb{R}$ such that $T_A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$? For $B$ as in P4, can you find $x, y \in \mathbb{R}$ such that $T_B \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$?

P6. For $A, B$ as in P4, find all vectors $(x, y)$ such that $T_A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 12 \end{pmatrix}$. Find all vectors $(x, y)$ such that $T_B \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 12 \end{pmatrix}$.


P8. For the matrix $A$ in P4, find vectors $(e, g), (f, h)$ such that $T_A \begin{pmatrix} e \\ g \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $T_A \begin{pmatrix} f \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Set $C = \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix}$. What is the composition $T_A \circ T_C$ as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$? What is the composition $T_C \circ T_A$?
P9. Consider the shear matrix \( S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). Compute \( T_\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( T_\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). What is the area of the image of the unit square under \( T_\theta \)? What is the area of the image of the rectangle with vertices \((0,0),(x,0),(0,y),(x,y)\)? What is the area of the image of the unit circle? Does \( T_\theta \) preserve the area of figures? Is \( T_\theta \) an isometry? Can you quickly write down the matrix \( U \) such that \( T_\theta \circ T_U = T_U \circ T_\theta = T_I \), where \( I \) is as in P4?

P10. Define the product of two matrices by

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}
\]

For the matrices in P4, P8, P9, what is \( AB, BA, AC, CA, SU, US \). How does \( AB \) compare with the matrix of compositions \( TA \circ TB \)? Do the same for \( AB \) and \( TB \circ TA \).

P11. PODASIP: The matrix of translation by the vector \((2,3)\) is \( \begin{pmatrix} 3 & 2 \\ 3 & 4 \end{pmatrix} \).

P12. PODASIP: The set of \( 2 \times 2 \) matrices is a group under matrix multiplication.

P13. PODASIP: For any \( 2 \times 2 \) matrix \( A \) and any vector \( \begin{pmatrix} a \\ b \end{pmatrix} \), the vectors

\[
T_A \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad T_A \begin{pmatrix} a \\ 0 \end{pmatrix} \quad T_A \begin{pmatrix} 0 \\ b \end{pmatrix} \quad T_A \begin{pmatrix} a \\ b \end{pmatrix}
\]

form the vertices of a rectangle.

Examples of Groups

P14. A subgroup \( H \) of a group \( G \) is a subset \( H \subset G \) which is itself a group with respect to \( G \)’s operation. Show that the even integers are a subgroup of \((\mathbb{Z},+), \) but the odd integers are not. Find all the subgroups of \((\mathbb{Z}_n,+)\).

P15. Show that the complex numbers of length one are a subgroup of \( \mathbb{C}^* = \mathbb{C} - \{0\} \) with respect to multiplication, but that the nonzero complex numbers of length at most one are not. (The length of a complex number \( z = x + iy \) is just its distance from the origin: \( |z| = \sqrt{x^2 + y^2} \).)

P16. Let \( \theta \) be a fixed angle and let \( R_\theta \) be rotation by angle \( \theta \) counterclockwise around the origin. Let \( R_\theta^n \) denote \( R_\theta \circ R_\theta \circ \ldots \circ R_\theta \). What is \( R_\theta^n \) geometrically? Define \( R_\theta^n \) similarly for \( n \in \mathbb{Z}^+ \). Let \( R_\theta^{-1} \) denote rotation by angle \( \theta \) clockwise around the origin, \( R_\theta^{-2} = R_\theta^{-1} \circ R_\theta^{-1} \), etc. Show that \( \{R_\theta^n \mid n \in \mathbb{Z}\} \) is a group with respect to composition. For which angles is this group finite? For which angles is this group infinite?

P17. PODASIP: The set of rotations around a fixed point in \( \mathbb{R}^2 \) is a subgroup of the group of isometries of \( \mathbb{R}^2 \). The set of translations in \( \mathbb{R}^2 \) is a subgroup. The set of all rotations around all points in \( \mathbb{R}^2 \) is a subgroup. The set of all translations and all rotations around all points in \( \mathbb{R}^2 \) is a subgroup. The set of reflections across all lines is a subgroup.

P18. PODASIP: Let \( S \) be a finite set. The set \( \mathcal{F}(S) \) of all functions \( f : S \to S \) is a group with respect to composition of functions (i.e. the operation on \( f,g \in \mathcal{F}(S) \) is \( f \circ g \)).

A Challenge Problem

P19 Say you are given two line \( \ell_1, \ell_2 \) on a piece of paper that are “almost parallel” – i.e. these lines intersect, but at a point \( O \) which is not on the paper. Your only tool is a straightedge. Given a point \( P \) on the paper and not on \( \ell_1 \) or \( \ell_2 \), how can you draw a line \( \ell_3 \) on the paper so that \( O, P \in \ell_3 \)?