Geometry and Symmetry, Problem Set #4
Summer 2009

Relating Synthetic and Analytic Geometry via the Number Line

*P1. (Warning: this is more a project than a single problem. Don't turn this in!) Using Hilbert's axioms alone, prove that the medians of a parallelogram bisect each other, i.e. the two pieces of the median are congruent. In addition to Hilbert's axioms, assume (i) a line ℓ is a set of points (so Hilbert's undefined *in* becomes $\in$), and (ii) for one line ℓ_0 there is a bijection $f : \ell_0 \to \mathbb{R}$ such that if A, B, C are any points on ℓ_0 with B between A and C, then $f(A) < f(B) < f(C)$. Is this enough to translate any proof of a theorem in synthetic geometry into a proof in analytic geometry?

Analytic geometry has a notion of distance. In addition to Hilbert's axioms, now assume that (i) a line ℓ is a set of points with a distance function $d : \ell \times \ell \to \mathbb{R}$ (i.e. $d(P,Q) = d(Q,P) \geq 0$, $d(P,Q) = 0$ iff $P = Q$, and triangle inequality holds), and (ii) for one line ℓ_0 there is an isometry $f : \ell_0 \to \mathbb{R}$. Is this enough to translate any analytic proof of a theorem involving distance into a proof that only involves Hilbert's axioms and the two assumptions above? How would we have to modify this process to translate analytic proofs about angles over to synthetic geometry? What about the same question for analytic proofs about areas?

Isometries of $\mathbb{R}^2$ – Numericals

Recall that every isometry f of $\mathbb{R}^2$ has the decomposition $f = T \circ R \circ \mathcal{F}$, where T is a translation, R is a rotation around the origin, and $\mathcal{F}$ is either the identity or the flip on the x-axis.

P2. What are $T, R, \mathcal{F}$ if f is the reflection across the line $y = 2x$?

P3. What if f is reflection across the line $y = 2x + 3$? *Hint:* Show that $f = T_{(0,3)} \circ g \circ T_{(0,-3)}$, where g is the (relabeled) reflection in P2. Now use the product formula for isometries from class.

P4. What if f is rotation by $\pi/3$ (counterclockwise, of course) around the point $(4, -1)$?

P5. What if f is translation by $(2, 1)$ followed by rotation by $\pi/4$ around the point $(1, 1)$?

P6. What if f is translation by $(3, 0)$ followed by reflection across the line $y = 2x$?

P7. What if f is the "glide reflection" given by flipping over the x-axis and then translating by $(4, 0)$?

P8. What if f is the "glide reflection" given by flipping over the line $y = -3x + 2$ and then translating by the vector $(1, -3)$ (which is parallel to the line)?

P9. A *fixed point* of a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a point P with $f(P) = P$. Which of the isometries in P2 – P8 have fixed points?

Synthetic Geometry and Projective Spaces

For each of the synthetic geometries below, which of the following properties hold?

A. *There is a unique line containing any pair of distinct points.*

B. *Any two distinct lines intersect at a unique point.*

C. *Given a line ℓ_1 and a point P not on ℓ_1, there exists a unique line ℓ_2 through P parallel to ℓ_1.*

P10. $\mathbb{RP}^2$, where the points of $\mathbb{RP}^2$ are the lines through the origin in $\mathbb{R}^3$ *except for the origin,* and the lines are the planes through the origin (except for the origin) in $\mathbb{R}^3$.

P11. $\mathbb{CP}^2$, with the definitions of points and lines the same as in P11, but with all coordinates complex numbers. (Recall that a line through the origin in $\mathbb{C}^3$ is $\{\lambda\vec{v} : \lambda \in \mathbb{C}\}$, and a plane through the origin is $\{\lambda\vec{w} + \mu\vec{z} : \lambda, \mu \in \mathbb{C}\}$ for some vectors $\vec{v}, \vec{w}, \vec{z} \in \mathbb{C}^3$.)

P12. $\mathbb{Z}_3\mathbb{P}^2$, with the definition of points and lines as in P11, but with all coordinates in $\mathbb{Z}_3$.

Matrices and Linear Transformations

(!)P13. For the moment, let's distinguish between a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and its associated *linear transformation* $T_A : \mathbb{R}^2 \to \mathbb{R}^2$, where

$$T_A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

Show that

$$T_A \circ T_B = T_{AB},$$

where AB is the usual product of matrices. This is precisely why matrix multiplication is defined as in Set 3, P10: matrix multiplication corresponds to composition of linear transformations.

P14. For T_A as in P13, show that (i) $T_A(\vec{v} + \vec{w}) = T_A(\vec{v}) + T_A(\vec{w})$ and (ii) $T_A(\lambda\vec{v}) = \lambda T_A(\vec{v})$ for all vectors $\vec{v}, \vec{w} \in \mathbb{R}^2$ and all $\lambda \in \mathbb{R}$. (Compare with Set 3, P4.) Functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ (or more generally from $\mathbb{R}^n$ to $\mathbb{R}^m$) with these two properties are called *linear transformations.*

P15. For each of the following statements about functions $f : \mathbb{R}^2 \to \mathbb{R}^2$, decide which ones imply which other ones. If one statement does not imply another, try to find and prove a salvage.

(a) $f = T_A$ for some 2×2 matrix A.

(b) f takes lines in $\mathbb{R}^2$ to lines in $\mathbb{R}^2$. (This is the origin of the name "linear transformation.")

(c) f has the properties (i), (ii) of P14.

Hint: Write lines as $\vec{v} + \lambda\vec{w}$.

P16. The map $T : A \mapsto T_A$ is a map from the set of 2×2 matrices to the set of functions on $\mathbb{R}^2$. Show that T is injective. Show that $(T_A \circ T_B) \circ T_C = T_A \circ (T_B \circ T_C)$. Use P13 to conclude that $(AB)C = A(BC)$, so we've checked that matrix multiplication is associative without computing the products of these three matrices.

Exploration

P17. Consider the circle $\mathcal{C}_r = \{(x, y) : x^2 + y^2 = r^2\}$ in the plane, where r is irrational. Does this circle contain any rational points, i.e. points both of whose coordinates are rational numbers? What if r is rational? Can you find all rational points on $\mathcal{C}_1$?