Geometry and Symmetry, Problem Set #5
Summer 2009

Exploration

P1. Imagine you are seized with the urge to paint. Put coordinates on $\mathbb{R}^3$ so that your right eye is at the origin and your line of sight is down the positive y-axis. Close your left eye, like all good artists. Think of the plane $y = 1$ as a transparent sheet which will serve as your canvas. A point $p = (x, y, z)$ with $y > 1$ is transferred to your canvas by intersecting the line from the origin to p with the canvas. Which sets of points in $\mathbb{R}^3$ are painted as the same point on the canvas? Do lines in $\mathbb{R}^3$ get painted as lines? What other figures in $\mathbb{R}^3$ get painted as lines? Do circles in $\mathbb{R}^3$ get painted as circles? What do parallel lines in $\mathbb{R}^3$ look like on your canvas? All good artists can paint a row of vertical telephone poles of uniform height and situated a uniform distance apart along a line in $\mathbb{R}^3$. On your canvas, the poles will be painted with different heights and different spacings. What are these heights and spacings?

Matrices

P2. For two 2×2 matrices A, B, show by a direct calculation that $\det(AB) = \det(A)\det(B)$. Conclude that if A has an inverse, then $\det(A) \neq 0$.

P3. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, try to solve $AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ directly. Conclude that A^{-1} exists iff $\det(A) \neq 0$.

P4. If A, B are invertible 2×2 matrices, show that $AB = I$ iff $BA = I$. Does this hold in any group?

P5. The *image* (or *range*) of a 2×2 matrix A is by definition $\{\vec{v} \in \mathbb{R}^2 : \vec{v} = A\vec{w} \text{ for some } \vec{w} \in \mathbb{R}^2\}$. PODASIP: The image of any matrix is either the origin, a line on the origin, or the entire plane $\mathbb{R}^2$.

P6. State and prove the analogue of P5 for 3×3 matrices $A : \mathbb{R}^3 \to \mathbb{R}^3$.

Isometries of $\mathbb{R}^2$

P7. Prove by example that an isometry is not determined two points: i.e. if f and g are isometries of $\mathbb{R}^2$ with $f(P) = g(P), f(Q) = g(Q)$ for distinct points P, Q, then it does not follow that $f = g$.

P8. PODASIP: An isometry of $\mathbb{R}^2$ is determined by three points: i.e. if f and g are isometries of $\mathbb{R}^2$ with $f(P) = g(P), f(Q) = g(Q), f(R) = g(R)$ for distinct points P, Q, R, then $f = g$. *Hint:* Show that $g^{-1} \circ f$ fixes P, Q, R. Can you show that for any other point W, $g^{-1} \circ f(W) = W$?

P9. PODASIP: An isometry of $\mathbb{R}^2$ which has no fixed points is a translation.

P10. PODASIP: An isometry of $\mathbb{R}^2$ which has more than one fixed point is either the identity or a reflection.

Elementary Properties of Groups

P11. What is the group of symmetries of an equilateral triangle? What are all the subgroups of this group?

The *order* of an element g in a group G is the smallest positive integer n such that $g^n = e$, where e is the identity in the group and g^n means the n-fold product $g \cdot g \cdot \ldots \cdot g$. (If no such n exists, we say that g has infinite order.)

P12. Find the order of every element of the group of symmetries of the triangle. Find the order of every element of the group of symmetries of the square. Find the order of every element of $(\mathbb{Z}, +)$. Find the order of every element of $\mathbb{Z}_2 \times \mathbb{Z}_3$ (the operation is addition on both groups). Any conjectures?

P13. PODASIP: The identity element in a group is unique. *Hint:* if $e, e' \in G$ are identities, what is ee'?

P14. PODASIP: If $ab = e$, then $ba = e$ in any group G with identity e.

P15. PODASIP: $(ab)^{-1} = a^{-1}b^{-1}$ in any group G.

P16. PODASIP: The set $\{\pm 1, \pm i\} \subset \mathbb{C}$ is a group with respect to multiplication which is isomorphic to $\mathbb{Z}_4$ with respect to addition.

P17. PODASIP: $\mathbb{Z}_8 \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$. (All operations are addition.)

P18. PODASIP: The group of symmetries of a equilateral triangle is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3$.

P19. PODASIP: A group cannot be isomorphic to a nontrivial subgroup of itself.

A Strange Function

P20. We showed in class that a bijective function $f : R^2 \to \mathbb{R}^2$ with the two properties of a linear transformation [(i)$f(x + y) = f(x) + f(y)$, (ii) $f(\lambda x) = \lambda f(x)$ for $\lambda \in \mathbb{R}$] is given by a 2×2 matrix. The analogue for bijective linear transformations $f : \mathbb{R} \to \mathbb{R}$ is a 1×1 matrix (a), so $f(x) = ax$. In this problem we show that a bijective function $f : \mathbb{R} \to \mathbb{R}$ with (i) only can be very complicated.

Play a multiplicative analogue of the UN game: for $x, y \in \mathbb{R}^* = \mathbb{R} - \{0\}$, set $x \sim y$ if $x/y \in \mathbb{Q}$. Show that this is an equivalence relation, and form a UN for this relation. If x_0 is in the UN, set $f(x_0) = y_0$, where $x_0 \sim y_0$ but $x_0 \neq y_0$. Make your choice of y_0 "random," i.e. avoid rules like $y_0 = 2x_0$ for all x_0. Now define f for any $x \in \mathbb{R}$ by demanding that (i) holds and that (ii) holds for any $\lambda \in \mathbb{Q}$ (not $\lambda \in \mathbb{R}$!). Prove that f is a bijection and has rule (i). Can you graph f?

Hyperbolic Finite Geometries?

*21. Is there a finite affine plane which is hyperbolic? I.e. is there a finite affine plane such that for a line ℓ and a point $P \notin \ell$, there exists more than one line ℓ' on P with $\ell' \| \ell$?