Geometry and Symmetry, Problem Set #6
Summer 2009

Exploration – Hyperbolic Triangles

*P1. How can we show that the sum of the angles in a hyperbolic triangle is less than π (radians)? Since LFTs are conformal, show that we may assume that one vertex of the triangle is at $(0,1)$ and a second is at $(0,a_1)$ for some $a_1 > 0$. The other two sides of the triangles are arcs of circles $\mathcal{C}_1 : (x-x_1)^2+y^2 = r_1^2$, $\mathcal{C}_2 : (x-x_2)^2+y^2 = r_2^2$. We may assume $x_1^2+a_1^2 = r_1^2$ $x_2^2+1 = r_2^2$ (why?).

The Geometry of the Determinant

P2. Show that the area of the parallelogram determined by the vectors $\vec{v} = (a,c), \vec{w} = (b,d)$ is $|ad-bc|$ in two ways: (i) draw a box around the parallelogram by drawing the lines $x = a+b, y = c+d$. Chop the area inside the box but outside the parallelogram up into nice boxes and triangles. Compute the area of the big box minus the area outside the parallelogram. (ii) The area is $|\vec{v}|\ |\vec{w}||\sin(\theta)|$, where θ is the angle between $\vec{v},\vec{w}$. Since $(b,d) \perp (d,-b)$, show that

$$|\vec{v}|\ |\vec{w}|\ |\sin(\theta)| = |\vec{v}|\ |(d,-b)|\ |\cos(\psi)| = |(a,c)\cdot(d,-b)| = |ad-bc|,$$

where ψ is the angle between $\vec{v}$ and $(d,-b)$.

P3. Conclude from P2 that the expansion/contraction of areas of rectangles under the matrix $A : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $|\det(A)|$. Show this implies $|\det(AB)| = |\det(A)|\ |\det(B)|$. This gives a geometric interpretation and proof of the multiplicativity of the determinant, but only up to sign.

P4. A is orientation preserving if the (counterclockwise) angle between $\vec{v} = A\begin{pmatrix}1\\0\end{pmatrix}$ and $\vec{w} = A\begin{pmatrix}0\\1\end{pmatrix}$ is in $(0,\pi)$, and orientation reversing if the angle is in $(\pi,2\pi)$. (Why don't we allow the angles 0 or π?) Check that this agrees with our previous thumb and index finger definition. Show that A is orientation preserving iff $\det(A) > 0$ and orientation reversing iff $\det(A) < 0$. Hint: If the angle between $\vec{v}$ and $\vec{w} = (b,d)$ is in $(0,\pi)$, then the angle ψ between $\vec{v}$ and $\vec{u} = (d,-b)$ is in $(-\pi/2,\pi/2)$. (Why?) Then

$$\det(A) = ad-bc = \vec{v}\cdot\vec{u} = |\vec{v}|\ |\vec{u}|\cos(\psi) > 0.$$

Give a similar proof that $\det(A) < 0$ if A is orientation reversing.

P5. Use P3 and P4 to prove $\det(AB) = \det(A)\det(B)$. Compare this proof to Set 5, P3. Which proof is harder? Which proof gives more insight?

P6. Let $SL(2,\mathbb{R})$ be the 2×2 matrices with real coefficients and determinant equal to one. Show that $SL(2,\mathbb{R})$ is a subgroup of $GL(2,\mathbb{R})$. Show that $SL(2,\mathbb{R})$ is precisely the set of linear transformations which preserve area. What is $SL(2,\mathbb{R}) \cap \mathrm{Isom}(\mathbb{R}^2)$?

The moral: The absolute value of the determinant measures the area scaling of the matrix. The sign of the determinant determines if the matrix is orientation preserving or reversing. The multiplicativity of the determinant is then geometrically obvious.

$GL(3, \mathbb{R})$ and $\mathbb{RP}^2$

P7. Show that the plane $z = 1$ in $\mathbb{R}^3$ is fixed by all matrices of the form $\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix}$. What is the effect of the following matrices on the $z = 1$ plane:

$$A = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix},\ C = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 5 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 4 \\ 0 & 0 & 1 \end{pmatrix},\ A{\cdot}B,\ B{\cdot}A,\ A^2,\ B^2$$

Do the results of this problem carry over if you replace $\mathbb{R}$ with $\mathbb{C}$ or $\mathbb{Z}_p$? (Compare with Set 3, P11.)

P8. When is a matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix}$ in $GL(3, \mathbb{R})$?

P9. What element of $A \in GL(3, \mathbb{R})$ acts on the $z = 1$ plane by translation $T_{(-2,3)}$? A is a symmetry of $\mathbb{RP}^2$. Think of the $z = 1$ plane as $\mathbb{R}^2$, and let $\alpha : \mathbb{R}^2 \to \mathbb{RP}^2$ be our usual embedding. You can either translate first and then embed, or embed first and then apply A. Show that you get the same result, so $\alpha(T_{(-2,3)}(P)) = A(\alpha(P))$. We say that A *extends* $T_{(-2,3)}$.

P10. Repeat P9 with the translation replaced by a rotation around the "origin" $(0, 0, 1)$ in the plane. Repeat P9 for a flip on the "x-axis" (the line $x = 0, z = 1$).

P11. Show that *isometries of* $\mathbb{R}^2$ *extend to symmetries of* $\mathbb{RP}^2$. Specifically, let f be an isometry of $\mathbb{R}^2$, thought of as the $z = 1$ plane. Then there exists a symmetry T of $\mathbb{RP}^2$ such that $T(\alpha(P)) = \alpha(f(P))$ for all $P \in \mathbb{R}^2$.

*P12 Show that (a group isomorphic to) $\mathrm{Isom}(\mathbb{R}^2)$ is a subgroup of $\mathrm{Symm}(\mathbb{RP}^2)$. (This is why I didn't want to impose a distance function on $\mathbb{RP}^2$ and look at isometries of $\mathbb{RP}^2$.)

Groups and Homomorphisms

All problems in this section are PODASIP.

P13. Groups have cancellation: i.e. if $a, b, g \in G$, then $ga = gb$ implies $a = b$. Similarly, if $ag = bg$, then $a = b$.

P14. Every group element appears exactly once in each row of a group's multiplication table. Every group elements appears exactly once in each column of a group's multiplication table.

P15. There is a unique group (up to isomorphism) of order two. Of order three. Of order four. Of order five.

A *homomorphism* $f : G \to H$ from a group G to a group H is a function satisfying $f(g_1 \cdot_G g_2) = f(g_1) \cdot_H f(g_2)$ for all $g_1, g_2 \in G$. (So a bijective homomorphism is an isomorphism.)

P16. The map $f : \mathbb{Z} \to \mathbb{Z}_n$ given by $f(a) = (a \bmod n)$ is a homomorphism (which is not an isomorphism). The map $h : \mathbb{Z}_n \to \mathbb{Z}$ given by $h(a \bmod n) = a$ is a homomorphism. The map $j : \mathbb{R} \to \mathbb{R}^2$ given by $j(x) = (x, 7)$ is a homomorphism. The map $k : \mathbb{R} \to \mathbb{R}^2$ given by $k(x) = (x, 2x)$ is a homomorphism. (All operations are addition.)

P17. Let $f : G \to H$ be a homomorphism of groups. Then $f(e_g) = e_H$, where e_G, e_H are the identity elements of G, H, respectively.

P18. Let $f : G \to H$ be a homomorphism of groups. Then for all $g \in G$, we have $f(g^{-1}) = (f(g))^{-1}$.

P19. Let $f : G \to H$ be a homomorphism of groups. The image of f is a subgroup of H.

P20. Let $f : G \to H$ be a homomorphism of groups, and let $h \in H$. Then the preimage of h (i.e. $\{g \in G : f(g) = h\}$) is a subgroup of G.

The Cross Ratio

Given points $a, b, c, d \in \mathbb{H}$, define the cross ratio to be $(a, b; c, d) = \frac{(a-c)/(a-d)}{(b-c)/(b-d)}$. The cross ratio may be undefined.

P21. Show that $(ai, i, 0, i\infty) = a$, i.e. $\lim_{b\to\infty}(ai, i; 0, ib) = a$.

P22. By considering the various cases, show that $(a, b; c, d) = (Aa, Ab; Ac, Ad)$ for any LFT A.

P23. Let $P, Q \in \mathbb{H}$ lie on a semicircle that hits the x-axis at M, N. Find the unique LFT A with $A(P) = i, A(M) = 0, A(N) = i\infty$. (Make sense out of this last condition.) Then $A(Q) = ai$ for some $a > 0$. Set $d_{\mathbb{H}}(P, Q) = |\ln(a)|$. Conclude that for all LFTs B, $d_{\mathbb{H}}(B(P), B(Q)) = d_{\mathbb{H}}(P, Q)$ for all $P, Q \in d_{\mathbb{H}}$. What happens if P, Q lie on a vertical line?

P24. Conclude that the group of LFTs is a subgroup of the group Isom($\mathbb{H}$) of isometries of $\mathbb{H}$ for $d_{\mathbb{H}}$. Is LFT = Isom($\mathbb{H}$)? Conclude also that all hyperbolic lines have infinite hyperbolic length.

P25. Properies of ln used above are (i) $\ln(1) = 0$, (ii) $\ln : \mathbb{R}^+ \to \mathbb{R}$ is an increasing surjection. (Why are (i), (ii) important?) How would the results of P21 - P24 change if we defined

$$d'_{\mathbb{H}}(ai, i) = \left|\tan\left(\frac{\pi a}{1+a} - \frac{\pi}{2}\right)\right|?$$