Geometry and Symmetry, Problem Set 7
Summer 2009

Exploration – Conic Sections

P1. Cut the cone $x^2 + y^2 = z^2$ with a plane $ax + by + cz = d, d \neq 0$ to get a curve $\mathcal{C}$. Show that the projection of $\mathcal{C}$ into the xy-plane is either an ellipse (possibly a circle), a hyperbola, a parabola, one line, or a point. Now show that $\mathcal{C}$ itself is one of these shapes (or possibly two lines). Restrict the plane to be tangent to S^2. Show that as you vary the cone to $x^2 + y^2 = \alpha z^2$, with $\alpha \in \mathbb{R}^+$, you get all ellipses, hyperbolas and parabolas as possible $\mathcal{C}$'s; equivalently, the projections are all (nonempty) graphs of the form $ax^2 + by^2 + cxy + dx + ey + f = 0$ for $a, b, c, d, e, f \in \mathbb{R}$. Conclude that there is a "$\beta^{-1} \circ \alpha$" map taking any ellipse, hyperbola or parabola to a circle. Finally, conclude that any coincidence theorem about an ellipse, hyperbola or parabola is true iff it holds for all circles.

Matrices acting on $\mathbb{R}^3$.

Set $\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

P2. Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & r \end{pmatrix}, B = \begin{pmatrix} a' & b' & c' \\ d' & e' & f' \\ g' & h' & r' \end{pmatrix}$ satisfy $AB = BA = Id$. Show that

$$A\vec{i} = \begin{pmatrix} a \\ d \\ g \end{pmatrix}, \quad A\vec{j} = \begin{pmatrix} b \\ e \\ h \end{pmatrix}, \quad A\vec{k} = \begin{pmatrix} c \\ f \\ r \end{pmatrix}$$

If we call these three vectors $\vec{v}, \vec{w}, \vec{q}$, show that

$$a'\vec{v} + d'\vec{w} + g'\vec{q} = \vec{i}, \; b'\vec{v} + e'\vec{w} + h'\vec{q} = \vec{j}, \; c'\vec{v} + f'\vec{w} + r'\vec{q} = \vec{k}.$$

P3. Show that a matrix $A \in \mathcal{M}_{3\times 3}$ is invertible iff none of the vectors $A\vec{i}, A\vec{j}, A\vec{k}$ lies in the plane formed by the other two iff $A : \mathbb{R}^3 \to \mathbb{R}^3$ is surjective iff $A : \mathbb{R}^3 \to \mathbb{R}^3$ is injective. *Hint:* Use P2. Compare with Set 5, P6.

Isometries of $\mathbb{R}^3$ – Food for Thought

P4. The distance between $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3$ is defined to be $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$. Isometries of $\mathbb{R}^3$ are by definition distance preserving maps $f : \mathbb{R}^3 \to \mathbb{R}^3$. Are translations isometries? Are reflections across a plane? Are rotations around some axis? Are their compositions? Supply proofs. Do we have them all? Rotations in $\mathbb{R}^2$ have a unique fixed point. Is there an isometry of $\mathbb{R}^3$ with a unique fixed point?

P5. Take a rotation R_1 around the z-axis by angle $\pi/4$ and a rotation R_2 around the x-axis by angle $\pi/4$. Is $R_1 \circ R_2 = R_2 \circ R_1$? Can you find a fixed point (other than $\vec{0}$) of $R_1 \circ R_2$? Of $R_2 \circ R_1$?

Projective Geometry

P6. Prove that every affine plane embeds in a projective plane. Prove that if a projective plane minus any line ℓ (and all the lines meeting ℓ) is an affine plane.

P7. Show that there is a perspectivity (i.e. a symmetry of $\mathbb{RP}^2$) taking a fixed line $\ell : ax+by = c$ to the line at infinity ℓ_∞. What is this perspectivity in (x, y)-coordinates as a map $\mathbb{R}^2 \setminus \ell \to \mathbb{R}^2$?

P8. Desargues' Theorem states that if ΔABC and $\Delta A'B'C'$ are in perspective from O, then $R = AB \cap A'B', S = BC \cap B'C', T = AC \cap A'C'$ are collinear. We gave a "3D" proof in class. Reprove Desargues' Theorem by moving (any line on) O to infinity. Give a third proof by moving RS to infinity, and then showing that T is on the line at infinity by a similar triangles argument. Give a fourth proof by moving RS to infinity and then giving a coordinate/analytic proof.

P9. Prove the converse of Desargues' Theorem by moving the line RST to infinity.

P10. PODASIP: Desargues' Theorem is true in $\mathbb{Z}_p^2$. (*Hint:* Can you give a 3D proof in $\mathbb{Z}_p^3$? Can you move any line to infinity? Can you reproduce the coordinate proof of P8?) What if we replace $\mathbb{Z}_p^2$ by $\mathbb{C}^2$? By $\mathbb{Q}(\sqrt{7})^2$? By k^2, where k is any field? (Look up the definition of a field.)

P11. PODASIP: Pappus' Theorem is true in $\mathbb{Z}_p^2$, in $\mathbb{C}^2$, in $\mathbb{Q}(\sqrt{7})^2$, in k^2 for any field k.

P12. Find the dual theorem to Desargues' Theorem. Do the same for Pappus' Theorem and Pascal's Theorem.

Affine and Projective Geometries

P13. If a line ℓ in an affine plane intersects a line m, and if m is parallel to the line n, prove that ℓ intersects n.

P14. If an affine plane has order n (i.e. every line contains n points), prove that there are at most $n + 2$ points such that no three are collinear.

P15. Prove that a projective plane of order greater than two has at least four lines with no three containing a common point.

P16. If a projective plane has order n, prove that every point lies on n lines.

Groups

P17. A subgroup H of a group G is *normal* if $ghg^{-1} \in H$ for all $g \in G, h \in H$. Find all the normal subgroups of the symmetry group of the equilateral triangle. Find all the normal subgroups of the symmetry group of the square.

P18. PODASIP: Let $f : G \to H$ be a group homomorphism. Then the *kernel* of f, $\text{Ker}(f) = \{g \in g : f(g) = e_h\}$ is a normal subgroup of G, and the image of f is a normal subgroup of H.

P19. Define the *center* of a group G to be $Z = Z(G) = \{z \in G : zg = gz \ \forall g \in G\}$. Show that Z is a normal subgroup of G. What does it mean if $Z = G$? Can $Z = \{e\}$? Find a group with Z a nontrivial subgroup of G. Find the centers of the symmetry groups of the equilateral triangle and the square.

P20. Let G be a group and H a subgroup. Define a relation on G by $g_1 \sim g_2$ iff $g_1^{-1}g_2 \in H$. Show that this is an equivalence relation. Define the *coset* (= country) of g to be $\{g_1 \in G : g_1 \sim g\}$. What is the coset of e? Of $h \in H$? Show that every $g \in G$ is an element of a unique coset. If $G = Z$ and $H = 17\mathbb{Z}$, build a UN (a set with one representative from each country). Can you do this for $G = \mathbb{R}$ and $H = \mathbb{Q}$? Show that the coset containing g equals $gH = \{gh : h \in H\}$ and conclude that all cosets are in one-to-one correspondence with H.

P21. Let g be an element of a group G. Let $\langle g \rangle = \{g^n : n \in \mathbb{Z}\}$ be the *cyclic group generated by g*. (By convention, $g^0 = e$.) Show that $\langle g \rangle$ really is a subgroup of G. What is $\langle g \rangle$ for $g \in G = U_{17}$ (the group of invertible elements in $\mathbb{Z}_{17}$)? What is $\langle g \rangle$ for $g \in G = U_{15}$? What is $\langle g \rangle$ for $g \in G = \mathbb{Z}$? Show that $\langle g \rangle$ is the smallest subgroup of G which contains g. Show that the order (= size) of $\langle g \rangle$ equals the order of the element g.

P22. Let H be a subgroup of a finite group G. Show that the order (= size) of H divides the order of G. (Hint: in P19, all the countries have the same order, namely the order of H.) Conclude from P20 that the order of an element divides the order of the group (Lagrange's Theorem).