Geometry and Symmetry, Problem Set 7
Summer 2015

Exploration – Conic Sections

P1. Cut the cone \( x^2 + y^2 = z^2 \) with a plane \( ax + by + cz = d \) to get a curve \( C \). Show that the projection of \( C \) into the \( xy \)-plane is either an ellipse (possibly a circle), a hyperbola, a parabola, a line, or a point. Now show that \( C \) itself is one of these shapes (or possibly two lines). Restrict the plane to be tangent to \( S^2 \). Show that as you vary the cone to \( x^2 + y^2 = \alpha z^2 \), with \( \alpha \in \mathbb{R}^+ \), you get all ellipses, hyperbolas and parabolas as possible \( C \); equivalently, the projections are all (nonempty) graphs of the form \( ax^2 + by^2 + cxy + dx + ey + f = 0 \) for \( a, b, c, d, e, f \in \mathbb{R} \). Note that any configuration of lines and an ellipse (or hyperbola or parabola) maps to a configuration in the \( z = 1 \) plane in which the ellipse (or hyperbola or parabola) is replaced by a circle. Conclude that any concordance theorem about an ellipse, hyperbola or parabola is true iff it holds for all circles. To prove such a theorem, check that we can move any line in the configuration to infinity via a \( \beta^{-1} \circ \alpha \) map.

Matrices acting on \( \mathbb{R}^3 \).

Set \( \vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \)

P2. Let \( A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & r \end{pmatrix} , B = \begin{pmatrix} a' & b' & c' \\ d' & e' & f' \\ g' & h' & r' \end{pmatrix} \) satisfy \( AB = BA = Id \). Show that

\[
A\vec{i} = \begin{pmatrix} a \\ d \\ g \end{pmatrix} , \quad A\vec{j} = \begin{pmatrix} b \\ e \\ h \end{pmatrix} , \quad A\vec{k} = \begin{pmatrix} c \\ f \\ r \end{pmatrix} .
\]

If we call these three vectors \( \vec{v}, \vec{w}, \vec{q} \), show that

\[
a'\vec{v} + d'\vec{w} + g'\vec{q} = \vec{i} , \quad b'\vec{v} + e'\vec{w} + h'\vec{q} = \vec{j} , \quad c'\vec{v} + f'\vec{w} + r'\vec{q} = \vec{k} .
\]

P3. Show that a matrix \( A \in M_{3 \times 3} \) is invertible iff none of the vectors \( A\vec{i}, A\vec{j}, A\vec{k} \) lies in the plane formed by the other two iff \( A : \mathbb{R}^3 \to \mathbb{R}^3 \) is surjective iff \( A : \mathbb{R}^3 \to \mathbb{R}^3 \) is injective. Hint: Use P2.

Isometries of \( \mathbb{R}^3 \) – Food for Thought

P4. The distance between \( (x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3 \) is defined to be

\[
\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} .
\]

Isometries of \( \mathbb{R}^3 \) are by definition distance preserving maps \( f : \mathbb{R}^3 \to \mathbb{R}^3 \). Are translations isometries? Are reflections across a plane? Are rotations around some axis? Are their compositions? Supply proofs. Do we have them all? Rotations in \( \mathbb{R}^2 \) have a unique fixed point. Is there an isometry of \( \mathbb{R}^3 \) with a unique fixed point?

P5. Take a rotation \( R_1 \) around the \( z \)-axis by angle \( \pi/4 \) and a rotation \( R_2 \) around the \( x \)-axis by angle \( \pi/4 \). Is \( R_1 \circ R_2 = R_2 \circ R_1 \)? Can you find a fixed point (other than \( \vec{0} \)) of \( R_1 \circ R_2 \)? Of \( R_2 \circ R_1 \)?
Projective Geometry

P6. Prove that every affine plane embeds in a projective plane. Prove that a projective plane minus any line \( \ell \) (and all the lines meeting \( \ell \)) is an affine plane.

P7. Show that there is a perspectivity (i.e. a symmetry of \( \mathbb{R}P^2 \)) taking a fixed line \( \ell : ax + by = c \) to the line at infinity \( \ell_\infty \). What is this perspectivity in \((x,y)\)-coordinates as a map \( \mathbb{R}^2 \setminus \ell \to \mathbb{R}^2 \)?

P8. Desargues’ Theorem states that if \( \triangle ABC \) and \( \triangle A'B'C' \) are in perspective from \( O \), then \( R = AB \cap A'B' \), \( S = BC \cap B'C' \), \( T = AC \cap A'C' \) are collinear. We gave a “3D” proof in class. Reprove Desargues’ Theorem by moving \( OT \) to infinity. Give a third proof by moving \( RS \) to infinity, and then showing that \( T \) is on the line at infinity by a similar triangles argument. Give a fourth proof by moving \( RS \) to infinity and then giving a coordinate/analytic proof.

P9. Prove the converse of Desargues’ Theorem by moving the line \( RST \) to infinity.

P10. PODASIP: Desargues’ Theorem is true in \( \mathbb{Z}_p^2 \). (Hint: Can you give a 3D proof in \( \mathbb{Z}_3^3 \)? Can you move any line to infinity? Can you reproduce the coordinate proof of P8?) What if we replace \( \mathbb{Z}_p^2 \) by \( C^2 \)? By \( \mathbb{Q}(\sqrt{7})^2 \)? By \( k^2 \), where \( k \) is any field? (Look up the definition of a field.)

P11. PODASIP: Pappus’ Theorem is true in \( \mathbb{Z}_p^2 \), in \( \mathbb{C}^2 \), in \( \mathbb{Q}(\sqrt{7})^2 \), in \( k^2 \) for any field \( k \).

P12. PODASIP: Desargues’ Theorem is true in any affine plane.

P13. Find the dual theorem for Pappus’ Theorem and for Pascal’s Theorem.

Affine and Projective Geometries

P14. If a line \( \ell \) in an affine plane intersects a line \( m \) (\( \ell \neq m \)) and if \( m \) is parallel to the line \( n \), prove that \( \ell \) intersects \( n \).

P15. Show that if one line in an affine plane has \( n \) points on it, then all lines have \( n \) points, and each point lies on \( n + 1 \) lines. Conclude that an affine plane of order \( n \) has \( n^2 \) points and \( n^2 + n \) lines, every line has \( n \) parallel lines (including itself), and that there are \( n + 1 \) sets of parallel lines.

P16. State and prove the corresponding result for projective planes.

Groups

P17. A subgroup \( H \) of a group \( G \) is normal if \( ghg^{-1} \in H \) for all \( g \in G \), \( h \in H \). Show that every subgroup of an abelian group is normal. Find all the normal subgroups of the symmetry group of the equilateral triangle. Find all the normal subgroups of the symmetry group of the square.

P18. PODASIP: Let \( f : G \to H \) be a group homomorphism. Then the kernel of \( f \), \( \text{Ker}(f) = \{ g \in g : f(g) = e_h \} \) is a normal subgroup of \( G \), and the image of \( f \) is a normal subgroup of \( H \).

P19. Define the center of a group \( G \) to be \( Z = Z(G) = \{ z \in G : zg = gz \ \forall g \in G \} \). Show that \( Z \) is a normal subgroup of \( G \). What does it mean if \( Z = G \)? Can \( Z = \{ e \} \)? Find a group with \( Z \) a nontrivial subgroup of \( G \). Find the centers of the symmetry groups of the equilateral triangle and the square.
P20. Let \( G \) be a group and \( H \) a subgroup. Define a relation on \( G \) by \( g_1 \sim g_2 \) if \( g_1^{-1}g_2 \in H \). Show that this is an equivalence relation. Define the coset (= country) of \( g \) to be \( \{g_1 \in G : g_1 \sim g\} \). What is the coset of \( e \)? Of \( h \in H \)? Show that every \( g \in G \) is an element of a unique coset. If \( G = \mathbb{Z} \) and \( H = 17\mathbb{Z} \), build a UN (a set with one representative from each country). Can you do this for \( G = \mathbb{R} \) and \( H = \mathbb{Q} \)? Show that the coset containing \( g \) equals \( gH = \{gh : h \in H\} \) and conclude that all cosets are in one-to-one correspondence with \( H \).

P21. Let \( g \) be an element of a group \( G \). Let \( \langle g \rangle = \{g^n : n \in \mathbb{Z}\} \) be the cyclic group generated by \( g \). (By convention, \( g^0 = e \).) Show that \( \langle g \rangle \) really is a subgroup of \( G \). What is \( \langle g \rangle \) for \( g \in G = U_{17} \) (the group of invertible elements in \( \mathbb{Z}_{17} \))? (Here the operation is multiplication in \( \mathbb{Z}_{17} \).) What is \( \langle g \rangle \) for \( g \in G = U_{15} \)? What is \( \langle g \rangle \) for \( g \in G = \mathbb{Z} \) with respect to addition? Show that \( \langle g \rangle \) is the smallest subgroup of \( G \) which contains \( g \). Show that the order (= size) of \( \langle g \rangle \) equals the order of the element \( g \).

P22. Let \( H \) be a subgroup of a finite group \( G \). Show that the order (= size) of \( H \) divides the order of \( G \). (Hint: in P20, all the countries have the same order, namely the order of \( H \).) Conclude from P21 that the order of an element divides the order of the group (Lagrange's Theorem).