Geometry and Symmetry, Problem Set 8  
Summer 2015

Exploration – Intersecting Polynomials in $\mathbb{C}P^2$

P1. Find the intersection of the following polynomials in $\mathbb{R}^2$, $\mathbb{C}$ and $\mathbb{C}P^2$: i.e. first consider $(x,y)$ as a point in $\mathbb{R}^2$ (except in (vi)), then in $\mathbb{C}$, then in $\mathbb{C}P^2$. You’ll have to make the equations homogeneous for points in $\mathbb{C}P^2$.

(i) $xy = 0$ and $xy = 1$
(ii) $y^2 = x^3 + x + 1$ and $y = 0$
(iii) $y^2 = x^3 + x + 1$ and $x = 0$
(iv) $y^2 = x^3 + x - 1$ and $x = 0$
(v) $x^2 + y^2 = 1$ and $(x - 3/2)^2 + y^2 = 1$
(vi) $y^2 = x^3 - 3x + 1$ and $y = i$.

Isometries of $\mathbb{R}^3$.

P2. Show that for $\bar{v}, \bar{w} \in \mathbb{R}^3$, we have $4\bar{v} \cdot \bar{w} = |ar{v} + \bar{w}|^2 - |ar{v} - \bar{w}|^2$. Conclude that a linear transformation/matrix $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is an isometry iff $\bar{v} \cdot \bar{w} = A\bar{v} \cdot A\bar{w}$ for all $\bar{v}, \bar{w} \in \mathbb{R}^3$.

Show that $A\bar{v} \cdot \bar{w} = \bar{v} \cdot A^T \bar{w}$, where the transpose matrix $A^T$ has $(i,j)^{\text{th}}$ entry equal to the $(j,i)^{\text{th}}$ entry of $A$.

P3. Let $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be an isometry with $A(\bar{0}) = \bar{0}$. Show that $A$ is continuous. Show that $A$ is injective and is given by a matrix. Show that $A\vec{i}, A\vec{j}, A\vec{k}$ cannot lie in a common plane. Conclude that $A$ is invertible (and so $A$ is surjective).

P4. PODASIP: Define the orthogonal group $O(3)$ to be the group of isometries of $\mathbb{R}^3$ preserving the origin. Then every isometry $f$ of $\mathbb{R}^3$ can be written as $f = T \circ A$, where $T$ is translation by $f(\bar{0})$ and $A \in O(3)$.

P5. PODASIP: $O(3) = \{A \in M_{3 \times 3} : A$ has columns which are mutually perpendicular and of length one$\}$. 

P6. PODASIP: $O(3) = \{A \in M_{3 \times 3} : A^T A = I\}$. Is $O(3) = \{A \in M_{3 \times 3} : AA^T = I\}$? Hint: Use P5.

P7. Let $R_x, \theta$ be the rotation in $\mathbb{R}^3$ around the $x$-axis by angle $\theta$ (counterclockwise with respect to the $y$- and $z$-axis), and similarly define $R_y, \theta, R_z, \theta$. Show that for each $A \in SO(3)$, there exist unique angles $\theta_1, \theta_2, \theta_3$ such that $A = R_{x, \theta_1} \circ R_{y, \theta_2} \circ R_{z, \theta_3}$. Hint: Rotate $\bar{k}$ around the $x$-axis until it becomes a vector $\bar{r}$ in the $xz$-plane. Rotate $\bar{k}$ around the $y$-axis until it coincides with $\bar{r}$. Conclude that there exist $\theta_1, \theta_2$ such that $A\bar{k} = R_{x, \theta_1} \circ R_{y, \theta_2} \bar{k}$. Note that $A\bar{k} = R_{x, \theta_1} \circ R_{y, \theta_2} \circ R_{z, \theta_3} \bar{k}$ for any $\theta_3$. Now find a $\theta_3$ such that $A\bar{j} = R_{x, \theta_1} \circ R_{y, \theta_2} \circ R_{z, \theta_3} \bar{j}$. Show that $A\bar{i} = R_{x, \theta_1} \circ R_{y, \theta_2} \circ R_{z, \theta_3} \bar{i}$ as well. (The $\theta_i$ are called the Euler angles of $A$.)

Determinants for $3 \times 3$ Matrices.

By an argument similar to P3, $A \in GL(3, \mathbb{R})$ iff the box/parallelepiped in $\mathbb{R}^3$ formed by $A\vec{i}, A\vec{j}, A\vec{k}$ is a real box – i.e. doesn’t lie in a plane. For $2 \times 2$ matrices, the determinant measures the area of the corresponding parallelogram in $\mathbb{R}^2$. So we want a determinant for $3 \times 3$ matrices which measures the volume of this box (at least up to sign).
P8. The area of the parallelogram formed by \( \vec{v}, \vec{w} \) with common angle \( \theta \) is \( |\vec{v}| |\vec{w}| \sin(\theta) \). (Why?) Show that this area is \( (\vec{v} \cdot \vec{w})(|\vec{w}|)^2)^{1/2} \). Conclude the Cauchy-Schwarz inequality: \( |\vec{v} \cdot \vec{w}| \leq |\vec{v}| |\vec{w}| \), with equality iff \( \vec{v} \) is a multiple of \( \vec{w} \). Reproduce Cauchy-Schwarz using \( \vec{v} \cdot \vec{w} = |\vec{v}| |\vec{w}| \cos(\theta) \).

P9. Let \( \vec{v} = (v_1, v_2, v_3), \vec{w} = (w_1, w_2, w_3) \in \mathbb{R}^3 \). Tediously compute that the area of the parallelogram formed by \( \vec{v}, \vec{w} \) is the length of the vector \( (v_2 w_3 - v_3 w_2) \hat{i} + (v_3 w_1 - v_1 w_3) \hat{j} + (v_1 w_2 - v_2 w_1) \hat{k} \). Show that this vector, denoted \( \vec{v} \times \vec{w} \), is perpendicular to both \( \vec{v}, \vec{w} \). Does this product make \( \mathbb{R}^3 \) into a group?

P10. Consider a parallelepiped formed by vectors \( \vec{v}, \vec{w}, \vec{s} \in \mathbb{R}^3 \). Show that the volume of the parallelepiped is \( |\vec{v} \times \vec{w}| |\vec{s}| \cos(\psi) \), where \( \psi \) is the angle between \( \vec{s} \) and \( \vec{v} \times \vec{w} \). Conclude that this volume is \( s \cdot (\vec{v} \times \vec{w}) \). Now tediously compute that this volume equals the absolute value of the determinant of \( A = \begin{pmatrix} v_1 & w_1 & s_1 \\ v_2 & w_2 & s_2 \\ v_3 & w_3 & s_3 \end{pmatrix} \), where \( \vec{s} = (s_1, s_2, s_3) \) and the determinant of this matrix is by definition

\[
\det(A) = v_1 w_2 s_3 + w_1 s_2 v_3 + s_1 v_2 w_3 - v_1 s_2 w_3 - w_1 v_2 s_3 - s_1 w_2 v_3.
\]

Thus the (absolute value of the) determinant of a \( 3 \times 3 \) matrix computes the volume of the parallelepiped.

P11. Show that a \( 3 \times 3 \) matrix \( B \) is invertible iff \( \det(B) \neq 0 \). Show that if \( B \) is an isometry, then \( \det(B) = \pm 1 \).

P12. Show that for \( 3 \times 3 \) matrices \( A, B, \det(AB) = \det(A) |\det(B)| \). Hint: Avoid a direct calculation by using P10. Conclude that the absolute value of the determinant is the volume scaling factor of the matrix. Compare with Set 6, P3.

P13. We say that an invertible \( 3 \times 3 \) matrix \( A \) is orientation preserving if you can place the middle finger, index finger and thumb along \( A \hat{i}, A \hat{j}, A \hat{k} \), respectively, without breaking your hand. If you have to break your hand, \( A \) is orientation reversing. Prove that \( A \) is orientation preserving iff \( \det(A) > 0 \). Hint: Generalize Set 6, P4.

P14. Show (without doing a direct tedious calculation) that for \( 3 \times 3 \) matrices \( A, B, \det(AB) = \det(A) \det(B) \). (Don’t forget that \( A \) or \( B \) might be noninvertible.)

Finite Geometries

P15. If an affine plane has order \( n \), prove that there are at most \( n + 2 \) points such that no three are collinear.

P16. Prove that a projective plane of order greater than one has at least four lines with no three containing a common point.

Groups

A subgroup \( H \) of a group \( G \) is a normal subgroup if \( h \in H, g \in G \) implies \( g^{-1}hg \in H \). (Why would we care about this?)

P17. PODASIP. \( \text{SO}(3) \) is a normal subgroup of \( \text{Isom}(\mathbb{R}^3) \).

P18. PODASIP. \( \text{SO}(3) \) is a normal subgroup of \( \text{O}(3) \).

P19. PODASIP. The group of translations of \( \mathbb{R}^3 \) is a normal subgroup of \( \text{Isom}(\mathbb{R}^3) \).
P20. PODASIP. Let $K, H, G$ be groups. If $K$ is a normal subgroup of $H$, and $H$ is a normal subgroup of $G$, then $K$ is a normal subgroup of $G$.

Challenge Problem

P21 Given two circles $C_1, C_2$ in the plane, find all circles tangent to both $C_1, C_2$. 