**Geometry and Symmetry, Problem Set 8**
**Summer 2009**

## Exploration – Intersecting Polynomials in $\mathbb{CP}^2$

P1. Find the intersection of the following polynomials in $\mathbb{R}^2, \mathbb{C}^2$ and $\mathbb{CP}^2$ : i.e. first consider $(x, y)$ as a point in $\mathbb{R}^2$ (except in (v)), then in $\mathbb{C}^2$, then in $\mathbb{CP}^2$. You'll have to make the equations homogeneous for points in $\mathbb{CP}^2$.

(i) $xy = 0$ and $xy = 1$

(ii) $y^2 = x^3 + x + 1$ and $y = 0$

(iii) $y^2 = x^3 + x + 1$ and $x = 0$

(iv) $y^2 = x^3 + x - 1$ and $x = 0$

(v) $y^2 = x^3 - 3x + 1$ and $y = i$.

## Isometries of $\mathbb{R}^3$.

P2. Show that for $\vec{v}, \vec{w} \in \mathbb{R}^3$, we have $4\vec{v} \cdot \vec{w} = |\vec{v} + \vec{w}|^2 - |\vec{v} - \vec{w}|^2$. Conclude that a linear transformation/matrix $A : \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry iff $\vec{v} \cdot \vec{w} = A\vec{v} \cdot A\vec{w}$ for all $\vec{v}, \vec{w} \in \mathbb{R}^3$. Show that $A\vec{v} \cdot \vec{w} = \vec{v} \cdot A^T\vec{w}$, where the transpose matrix $A^T$ has $(i, j)^{\text{th}}$ entry equal to the $(j, i)^{\text{th}}$ entry of $A$.

P3. Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be an isometry with $A(\vec{0}) = \vec{0}$. Show that $A$ is continuous. Show that $A$ is injective and is given by a matrix. Show that $A\vec{i}, A\vec{j}, A\vec{k}$ cannot lie in a common plane. Conclude that $A$ is invertible (and so $A$ is surjective).

P4. PODASIP: Define the *orthogonal group* $O(3)$ to be the group of isometries of $\mathbb{R}^3$ preserving the origin. Then every isometry $f$ of $\mathbb{R}^3$ can be written as $f = T \circ A$, where $T$ is translation by $f(\vec{0})$ and $A \in O(3)$.

P5. PODASIP: $O(3) = \{A \in \mathcal{M}_{3\times3} : AA^T = Id\}$. *Hint:* Use $A\vec{v} \cdot A\vec{w} = \vec{v} \cdot \vec{w}$.

P6. PODASIP: $O(3) = \{A \in \mathcal{M}_{3\times3} : A$ has columns which are mutually perpendicular and of length one$\}$.

P7. Let $R_{x,\theta}$ be the rotation in $\mathbb{R}^3$ around the $x$-axis by angle $\theta$ (counterclockwise with respect to the $y$- and $z$-axis), and similarly define $R_{y,\theta}, R_{z,\theta}$. Show that for each $A \in SO(3)$, there exist unique angles $\theta_1, \theta_2, \theta_3$ such that $A = R_{x,\theta_1} \circ R_{y,\theta_2} \circ R_{z,\theta_3}$. *Hint:* Rotate $A\vec{k}$ around the $x$-axis until it becomes a vector $\vec{r}$ in the $xz$-plane. Rotate $\vec{k}$ around the $y$-axis until it coincides with $\vec{r}$. Conclude that there exist $\theta_1, \theta_2$ such that $A\vec{k} = R_{x,\theta_1} \circ R_{y,\theta_2}\vec{k}$. Note that $A\vec{k} = R_{x,\theta_1} \circ R_{y,\theta_2} \circ R_{z,\theta_3}\vec{k}$ for any $\theta_3$. Now find a $\theta_3$ such that $A\vec{j} = R_{x,\theta_1} \circ R_{y,\theta_2} \circ R_{z,\theta_3}\vec{j}$. Show that $A\vec{i} = R_{x,\theta_1} \circ R_{y,\theta_2} \circ R_{z,\theta_3}\vec{i}$ as well. (The $\theta_i$ are called the Euler angles of $A$.)

## Determinants for $3 \times 3$ Matrices.

By an argument similar to P3, $A \in GL(3, \mathbb{R})$ iff the box/parallelipiped in $\mathbb{R}^3$ formed by $A\vec{i}, A\vec{j}, A\vec{k}$ is a real box – i.e. doesn't lie in a plane. For $2 \times 2$ matrices, the determinant measures the area of the corresponding parallelogram in $\mathbb{R}^2$. So we want a determinant for $3 \times 3$ matrices which measures the volume of this box (at least up to sign).

P8. The area of the parallelogram formed by $\vec{v}, \vec{w}$ with common angle $\theta$ is $|\vec{v}|\ |\vec{w}| \sin(\theta)$. (Why?) Show that this area is $[(\vec{v} \cdot \vec{v})(\vec{w} \cdot \vec{w}) - (\vec{v} \cdot \vec{w})^2]^{1/2}$. Conclude the Cauchy-Schwarz inequality: $|\vec{v} \cdot \vec{w}| \le |\vec{v}||\vec{w}|$, with equality iff $\vec{v}$ is a multiple of $\vec{w}$. Reprove Cauchy-Schwarz using $\vec{v} \cdot \vec{w} = |\vec{v}|\ |\vec{w}| \cos(\theta)$.

P9. Let $\vec{v} = (v_1, v_2, v_3), \vec{w} = (w_1, w_2, w_3) \in \mathbb{R}^3$. Tediously compute that the area of the parallelogram formed by $\vec{v}, \vec{w}$ is the length of the vector $(v_2w_3 - v_3w_2)\vec{i} + (v_3w_1 - v_1w_3)\vec{j} + (v_1w_2 - v_2w_1)\vec{k}$. Show that this vector, denoted $\vec{v} \times \vec{w}$, is perpendicular to both $\vec{v}, \vec{w}$. Does this product make $\mathbb{R}^3$ into a group?

P10. Consider a parallelepiped formed by vectors $\vec{v}, \vec{w}, \vec{s} \in \mathbb{R}^3$. Show that the volume of the parallelepiped is $|\vec{v} \times \vec{w}|\,|\vec{s}| \cos(\psi)$, where $\psi$ is the angle between $\vec{s}$ and $\vec{v} \times \vec{w}$. Conclude that this volume is $\vec{s} \cdot (\vec{v} \times \vec{w})$. Now tediously compute that this volume equals the absolute value of the determinant of $A = \begin{pmatrix} v_1 & w_1 & s_1 \\ v_2 & w_2 & s_2 \\ v_3 & w_3 & s_3 \end{pmatrix}$, where $\vec{s} = (s_1, s_2, s_3)$ and the determinant of this matrix is by definition

$$\det(A) = v_1w_2s_3 + w_1s_2v_3 + s_1v_2w_3 - v_1s_2w_3 - w_1v_2s_3 - s_1w_2v_3.$$

Thus the (absolute value of the) determinant of a $3 \times 3$ matrix computes the volume of the parallelepiped.

P11. Show that a $3 \times 3$ matrix $B$ is invertible iff $\det(B) \neq 0$. Show that if $B$ is an isometry, then $\det(B) = \pm 1$.

P12. Show that for $3 \times 3$ matrices $A, B$, $|\det(AB)| = |\det(A)|\,|\det(B)|$. *Hint:* Avoid a direct calculation by using P10. Conclude that the absolute value of the determinant is the volume scaling factor of the matrix. Compare with Set 6, P3.

P13. We say that that an invertible $3 \times 3$ matrix $A$ is *orientation preserving* if you can place the middle finger, index finger and thumb along $A\vec{i}, A\vec{j}, A\vec{k}$, respectively, without breaking your hand. If you have to break your hand, $A$ is *orientation reversing.* Prove that $A$ is orientation preserving iff $\det(A) > 0$. *Hint:* Generalize Set 6, P4.

P14. Show (without doing a direct tedious calculation) that for $3 \times 3$ matrices $A, B$, $\det(AB) = \det(A)\det(B)$. (Don't forget that $A$ or $B$ might be noninvertible.)

**Finite Geometries**

P15. Show that if one line in an affine plane has $n$ points on it, then all lines have $n$ points, and each point lies on $n+1$ lines. Conclude that an affine plane of order $n$ has $n^2$ points and $n^2 + n$ lines, every line has $n$ parallel lines (including itself), and that there are $n+1$ sets of parallel lines.

P16. State and prove the corresponding result for projective planes.

**Groups**

P17. PODASIP. SO(3) is a normal subgroup of Isom($\mathbb{R}^3$).

P18. PODASIP. The group of translations of $\mathbb{R}^3$ is a normal subgroup of SO(3).

P19. PODASIP. The group of translations of $\mathbb{R}^3$ is a normal subgroup of Isom($\mathbb{R}^3$).

P20. PODASIP. Let $K, H, G$ be groups. If $K$ is a normal subgroup of $H$, and $H$ is a normal subgroup of $G$, then $K$ is a normal subgroup of $G$.