

Geometry and Symmetry, Problem Set 9
Summer 2011

Exploration – Dictionary Groups

P1. We can write down every possible English word by listing the letters a, \dots, z of the alphabet and specifying word formation rules as follows: all letters are possible words, and given any two words w_1, w_2 , we can form a new word by the *concatenation product* $w_1 \star w_2$ which just puts w_1 next to w_2 . For example, $a \star b = ab, ab \star ab = abab, dog \star cat = dogcat$, etc. This product is clearly associative. We abbreviate words like $aaaaabbccdd$ by $a^5b^2cd^2$.

To make our dictionary into a group, we need an identity element, which can only be the “empty word”, denoted $_$, with the rule e.g. $dog \star _ = _ \star dog = dog$. We also need to add new letters $a^{-1}, b^{-1}, \dots, z^{-1}$ with the rules $a \star a^{-1} = a^{-1} \star a = \dots = z \star z^{-1} = z^{-1} \star z = _$. Now we have a group – for example, the inverse of cat is $t^{-1}a^{-1}c^{-1}$.

Let’s agree on a list of all possible swear words. We can impose the rule that if e.g. $abcde$ is a swear word, then we do not allow this word to be formed. In particular, we will expunge this word by declaring a new rule $abcde = a \star bcde = ab \star cde = abc \star de = abcd \star e = _$. Of course, if we eliminate $abcde$, we have to expunge its inverse as well. (Why?) Note that $hjkab \star cdem^3 = hjk \star abcde \star m^3 = hjkm^3$, so we delete swear words occurring in the middle of products. What group do we get if our alphabet consists only of the letter a (and it is understood that we also automatically have a^{-1}) and there are no swear words? What group do we get if our alphabet has only a , and a^{173} is a swear word? What group do we get if our alphabet has a, b and $aba^{-1}b^{-1}$ is a swear word? What group do we get if our alphabet has a, b and swear words a^2, b^3 ?

I claim that if you have a finite group $G = \{g_1, g_2, \dots, g_n\}$ and if you form a dictionary with the letters g_1, \dots, g_n and make every multiplication table entry a swear word, in the sense that $g_i g_j g_k^{-1}$ is a swear word if $g_i \cdot_G g_j = g_k$, then the group you get is isomorphic to G . Can you prove this? (Watch out for the “two” inverses for e.g. g_1 , namely its inverse g_j for some j and the new letter g^{-1} .)

Rotations in \mathbb{R}^3 and the quaternions

Let S^3 be the unit quaternions. Recall that we have a two-to-one homomorphism $\rho : S^3 \rightarrow \text{SO}(3)$ given by $\rho(q) = C_q, C_q(v) = qvq^{-1}$. (Here we think of $v \in \mathbb{R}^3$ as an imaginary quaternion.)

P2. Show that the axis of $\rho(q)$ is given by the imaginary part of q (thought of as a point in \mathbb{R}^3). *Hint:* Write $q = a + \text{Im}(q)$, with $a \in \mathbb{R}$ and $\text{Im}(q)$ imaginary (quaternionic). Show that $qqq^{-1} = q$ implies $q(\text{Im}(q))q^{-1} = \text{Im}(q)$. Conclude that $\rho(q)$ fixes $\text{Im}(q)$, so $\text{Im}(q)$ must be the axis.

P3. In the notation of P2, write $a = \cos(\theta/2)$ for some angle $\theta \in [0, 2\pi)$. Show that $\rho(q)$ is rotation by angle θ . (The angle depends on your choice of a unit vector on the axis.) *Hint:* We know the axis of $\rho(q)$ by P2, and we know that $\rho(q)$ is a rotation in a plane $\text{Im}(q)^\perp$ perpendicular to the axis. Take your favorite vector in $\text{Im}(q)^\perp$; if $\text{Im}(q) = bi + cj + dk$, my favorite is $\vec{v} = (-c, b, 0)$. Compute $\rho(q)\vec{v} = \vec{w}$ — it’s not that bad. Then the angle of rotation of $\rho(q)$ is given by $\vec{v} \cdot \vec{w} = |\vec{v}| |\vec{w}| \cos(\theta)$.

P4. Take $A, B \in \text{SO}(3)$, and say $\rho(q_1) = A, \rho(q_2) = B$. We know how to find q_1, q_2 , and we know that $\rho(q_1 q_2) = AB$. Conclude that the axis of AB is the imaginary part of $q_1 q_2$ and the angle of rotation of AB is determined by the real part of $q_1 q_2$. (This is by far the easiest way I know to find the axis and angle of the composition of rotations.)

- *P5. Let A be rotation by an irrational angle around the z -axis, and let B be rotation by the same angle around the x -axis. Show that the group generated by A and B is isomorphic to F_2 . Here F_2 is the free group on two generators, the dictionary group with two letters a, b (and their inverses) and no swear words except the obvious ones ($aa^{-1} = a^{-1}a = bb^{-1} = b^{-1}b = \underline{\quad}$). You have to show that no word in $A^{\pm 1}, B^{\pm 1}$ is the identity except for the obvious swear words. Try an induction on the number of letters in a word.
- P6. Find the center of S^3 .
- P7. Is $\{\pm 1\}$ a subgroup of S^3 ? Is it a normal subgroup?
- P8. Is $C = \{a + bi \mid |a|^2 + |b|^2 = 1\}$ an abelian subgroup of S^3 ? Is it isomorphic to a group we've seen many times? Is C a normal subgroup of S^3 ?

Quotient Groups

- P9. Show that every group G of order p is cyclic (i.e. there exists $g \in G$ such that $\langle g \rangle = G$). *Hint:* pick $g \neq e$. What does Lagrange's theorem tell you about $\langle g \rangle$?
- P10. Let H be a subgroup of a group G . Show that each left coset $gH = \{gh \mid h \in H\}$ equals the right coset $Hg = \{hg \mid h \in H\}$ iff $H \triangleleft G$ (i.e. H is a normal subgroup of G).
- P11. Let H be a subgroup of an abelian group G . Define a multiplication on (left) cosets by $g_1H \cdot g_2H = (g_1g_2)H$. Then the set of cosets (denoted G/H) is a group. *Hint:* the coset $H = hH$ for any $h \in H$ is the identity.
- P12. Let $17\mathbb{Z} = \{17k \mid k \in \mathbb{Z}\}$. Use P19 to show that $\mathbb{Z}/17\mathbb{Z} \simeq Z_{17}$.
- P13. Show that $\{0, 3\} \subset \mathbb{Z}_6$ is isomorphic to \mathbb{Z}_2 . Show that $\mathbb{Z}_6/\mathbb{Z}_2 \simeq \mathbb{Z}_3$, where \mathbb{Z}_2 stands for $\{0, 3\}$.
- P14. PODSIP. If A and B are abelian groups, then $G = A \times B$ is abelian. The subset $A_1 = A \times \{e_b\}$ is a subgroup of G isomorphic to A , and similarly for $B_1 = \{e_a\} \times B$. We have $G/A_1 \simeq B$ and $G/B_1 \simeq A$. (Of course everyone just denotes A_1, B_1 by A, B , respectively, so e.g. $(A \times B)/A \simeq B$, as it should.)
- P15. PODASIP. If $H \triangleleft G$, then G/H has a group structure given by $g_1H \cdot g_2H = (g_1g_2)H$.
- P16. Let $f : \mathbb{Z} \rightarrow \mathbb{Z}_{17}$ be $f(a) = (a \bmod 17)$. What is the kernel $K = \text{Ker}(f)$ of f ? Show that f induces a well defined map $\bar{f} : \mathbb{Z}/(\text{Ker}(f)) \rightarrow \mathbb{Z}_{17}$ by $\bar{f}(nK) = f(n)$.
- P17. Let A, B be groups, and let $f : A \times B \rightarrow A$ be $f(a, b) = a$. Show that f is a homomorphism. Compute the kernel of f . Find a group isomorphic to $(A \times B)/(\text{Ker}(f))$.
- P18. Prove the *First Isomorphism Theorem* of group theory: if $f : G \rightarrow H$ is a homomorphism of groups, then $G/\text{Ker}(f) \simeq \text{Im}(f)$. *Hint:* show that f induces a well defined homomorphism $\bar{f} : G/\text{Ker}(f) \rightarrow \text{Im}(f)$ by $\bar{f}(gK) = f(g)$. Show that \bar{f} is the isomorphism we want.

The shape of S^3

- P19. Recall $S^3 = \{(a, b, c, d) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + w^2 = 1\}$ is the unit 3-sphere in \mathbb{R}^4 . Writing $z = a + ib, w = c + id$, we can rewrite $S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$. Show that just as S^2 minus a point is homeomorphic to \mathbb{R}^2 (i.e. there is a continuous bijection from S^2 minus a point to \mathbb{R}^2 with continuous inverse), S^3 minus a point is homeomorphic to \mathbb{R}^3 .

- P20. Of course, $\mathbb{R}^2 = \mathbb{C}$ can be divided into two sets, $A = \{z : |z| \leq 1\}$ and $B = \{z : |z| \geq 1\}$, with $A \cap B$ the circle S^1 . A looks like a disk, and adding a “point at infinity” makes B a disk. (Why?) Thus S^2 can be divided into two disks A and B , glued along their common boundary S^1 . (Draw a picture.) Similarly, S^3 can be divided into two sets, $A = \{(z, w) : |z|^2 + |w|^2 = 1, |z| \leq |w|\}$, and $B = \{(z, w) : |z|^2 + |w|^2 = 1, |z| \geq |w|\}$. Show that $A \cap B$ is a torus. Show that A and B are homeomorphic to the solid torus $D^2 \times S^1$ glued along their boundary torus. (Here D^2 is the disk.) Conclude that \mathbb{R}^3 is given by gluing two solid tori along their boundaries and then removing one point.
- P21. Draw a picture of \mathbb{R}^3 as the union of two solid tori minus a point. Draw one solid torus in the standard way. How can you draw the second torus? Which lines of the second torus glue to meridian lines of the first torus, and which lines of the second torus glue to longitude lines of the first torus?